SURFACES ASSOCIATED TO ZEROS OF AUTOMORPHIC L-FUNCTIONS

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ABSTRACT. Assuming the Riemann Hypothesis, Montgomery established results concerning pair correlation of zeros of the Riemann zeta function. Rudnick and Sarnak extended these results to automorphic L-functions and all level correlations. We show that automorphic L-functions exhibit additional geometric structures related to the correlation of their zeros. In the case of pair correlation, these structures form certain surfaces which display Gaussian behavior. For triple correlation, these structures reveal characteristics of the Laplace and Chi-squared distributions.

1. INTRODUCTION

1.1. **Motivation.** In his seminal work [40], Montgomery conducted a detailed study of the pair correlation of non-trivial zeros of the Riemann zeta function $\zeta(s)$. To understand the Fourier transform of the distribution function of the numbers $\gamma-\gamma'$, where γ, γ' are imaginary parts of the non-trivial zeros of $\zeta(s)$, Montgomery considered the function

$$F(\alpha) := F(\alpha, T) = \left(\frac{T}{2\pi}\log\frac{T}{2\pi}\right)^{-1} \sum_{0<\gamma'\leqslant T}\sum_{0<\gamma\leqslant T} T^{i\alpha(\gamma-\gamma')} w(\gamma-\gamma'). \tag{1.1}$$

Here $\alpha \in \mathbb{R}$, $T \geqslant 2$ and $w(u)$ is a suitable weight function defined on $\mathbb{R}$ by $w(u) = 4/(4+u^2)$. Assuming the Riemann Hypothesis (RH) for $\zeta(s)$, Montgomery showed that

$$F(\alpha, T) = (1+o(1))T^{-2\alpha}\log T + \alpha + o(1), \tag{1.2}$$

as T tends to infinity, uniformly for $0 \leqslant \alpha < 1$.

Rudnick and Sarnak [47] vastly generalized Montgomery's work by establishing corresponding results for the higher level correlations of non-trivial zeros of general automorphic L-functions. Their work shows that the local fluctuations of the zeros of an automorphic L-function align precisely with the predictions of the GUE model as suggested by Dyson [8].

We discover additional geometric structures in the form of surfaces related to the correlation of zeros of general automorphic L-functions (see Figures 1–3). Let $m \in \mathbb{N}$ and $\mathscr{A}_m$ be the set of irreducible cuspidal automorphic representations of GL_m over $\mathbb{Q}$ with unitary central character. For $\pi \in \mathscr{A}_m$, let $L(s, \pi)$ be its standard L-function. For $X, T \geqslant 2$, we consider the sum

$$\mathcal{S}_{\mathrm{av},\pi}(X,T) = \frac{1}{N_\pi(T)} \sum_{|\mathrm{Im}\,\rho_\pi'|\leqslant T} \mathrm{Re} \sum_{|\mathrm{Im}\,\rho_\pi|\leqslant T} X^{\rho_\pi-\rho_\pi'} w(\rho_\pi-\rho_\pi'), \tag{1.3}$$

where

$$w : \mathbb{C}\setminus\{2,-2\} \to \mathbb{C}, \quad w(u) := \frac{4}{4-u^2}, \tag{1.4}$$

$$N_\pi(T) = \#\left\{\rho_\pi = \beta_\pi + i\gamma_\pi \colon 0<\beta_\pi<1, |\gamma_\pi| \leqslant T, L(\rho_\pi, \pi) = 0\right\}, \tag{1.5}$$

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and ρ_π, ρ'_π vary over the non-trivial zeros of $L(s,\pi)$. Here the function $w(u)$ in (1.4) corresponds to the weight function introduced by Montgomery in (1.1). We will discuss a more general class of weight functions in Section 7. We rewrite $\mathcal{S}_{\text{av},\pi}(X,T)$ as

$$\mathcal{S}_{\text{av},\pi}(X,T) = \frac{1}{N_\pi(T)} \sum_{|\text{Im}\,\rho'_\pi| \leqslant T} \text{Re}\,\mathcal{S}_\pi(X,T,\rho'_\pi) \tag{1.6}$$

where

$$\mathcal{S}_\pi(X,T,\rho'_\pi) = \sum_{|\text{Im}\,\rho_\pi| \leqslant T} X^{\rho_\pi - \rho'_\pi} w(\rho_\pi - \rho'_\pi). \tag{1.7}$$

When $\pi \in \mathscr{A}_1$ is trivial, assuming RH for $\zeta(s)$, one may consider $S_\pi(X,T,\rho'_\pi)$ as the contribution to $F(\alpha)$ from the inner sum in (1.1) corresponding to the zero ρ'_π when $X = T^\alpha$. In the present paper, we are interested in the behavior of $S_\pi(X,T,\rho'_\pi)$ as ρ'_π varies over the non-trivial zeros of $L(s,\pi)$. Surprisingly, numerical evidence, as shown in Figure 1, suggests that in the case of $\zeta(s)$, the associated surface appears to reveal Gaussian behavior led along Montgomery's function $F(\alpha)$. Our first aim in this paper is to study the precise nature of this surface for the Riemann zeta function. Our second goal is to extend the results in the broader context of automorphic L-functions. Our third objective is to investigate whether there is any change in these phenomena as one transitions from pairs to triples of zeros of automorphic L-functions. For further numerical and graphical evidence in these directions, interested readers are referred to [1].

1.2. **A Brief Survey of Montgomery's Pair Correlation Conjecture.** The function $F(\alpha)$ has been the subject of extensive research over the years. Besides proving (1.2), in [40], Montgomery also presented heuristics suggesting that

$$F(\alpha,T) = 1 + o(1) \quad \text{as} \quad T \to \infty, \tag{1.8}$$

uniformly for $\alpha \in [a,b]$, where $1 \leqslant a < b < \infty$ are any fixed constants. Assuming that (1.8) holds, it would follow that for any fixed $0 < \alpha < \beta < \infty$,

(1.9)
$$\frac{\#\left\{(\gamma,\gamma') : 0 < \gamma,\gamma' \leqslant T, 2\pi\alpha(\log T)^{-1} \leqslant \gamma - \gamma' \leqslant 2\pi\beta(\log T)^{-1}\right\}}{T(\log T)/(2\pi)} \sim \int_\alpha^\beta \left(1 - \left(\frac{\sin \pi u}{\pi u}\right)^2\right) \mathrm{d}u$$

as $T \to \infty$, where γ, γ' vary over the imaginary parts of the non-trivial zeros of Riemann zeta function $\zeta(s)$. The asymptotic formula (1.9), known as Montgomery's Pair Correlation conjecture, suggests that small gaps between zeros of $\zeta(s)$ occur very rarely, which is remarkable. Indeed, the pair correlation for many other number theoretic quantities differ from (1.9). For example, although the sequence of zeros of $\zeta(s)$ and the sequence of prime numbers are related by explicit formulas, their distributions are very different. More precisely, assuming an appropriate version of the prime k-tuple conjecture, Gallagher [11] showed that the sequence of primes in short intervals exhibits Poisson behavior.

Dyson [40] made the significant observation that the Gaussian Unitary Ensemble (GUE) from Random Matrix Theory has the same pair correlation function $1 - (\frac{\sin \pi u}{\pi u})^2$ as in the case of zeros of $\zeta(s)$. In mathematical physics, GUE models the distribution of energy levels in systems comprising numerous particles. When suitably normalized, the limiting pair correlation function of eigenvalues of matrices in the Gaussian Unitary Ensemble becomes $1 - (\frac{\sin \pi u}{\pi u})^2$. This observation, along with Odlyzko's [44, 45] extensive computations show strong evidence towards the Hilbert–Pólya conjecture that non-trivial zeros of the Riemann zeta function correspond to eigenvalues of a self-adjoint operator. For more results related to Montgomery's Pair Correlation conjecture, the reader is referred to the classical works of Gallagher [12], Goldston [14], Goldston–Montgomery [15], Goldston–Heath-Brown [20], Heath-Brown [18], Montgomery–Odlyzko [41] and the references there-in.

Rudnick and Sarnak [47] established the higher level correlations of non-trivial zeros of any given automorphic L-function. Their work shows that the local fluctuations of the zeros of an automorphic L-function are universal and independent of the distribution of its coefficients. Furthermore, these fluctuations align

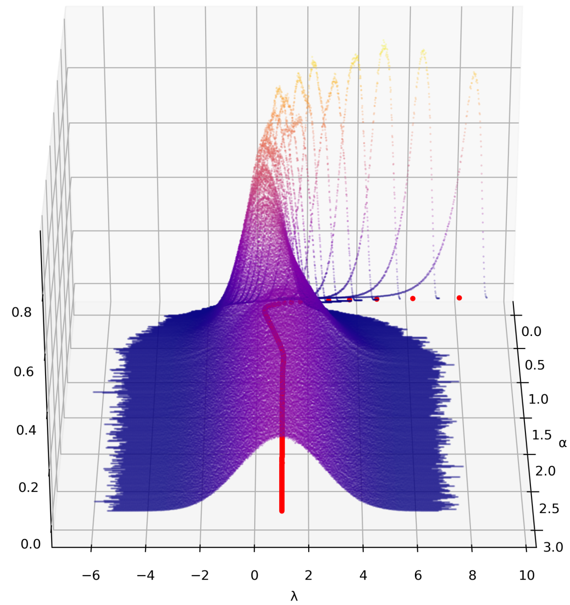

FIGURE 1. A surface plot for the probability density function associated to the set of values $\{\operatorname{Re} S_\pi(X,T,\rho_n)\colon 1 \leqslant n \leqslant N\}$, where $L(s,\pi)=\zeta(s)$, $N=10^6$, $X=T^\alpha$ and α varies between $(0,3]$ in a discrete equi-spaced manner. The surface (in blue) appears to exhibit Gaussian behavior while shifting along Montgomery's function $F(\alpha)$ (in red).

precisely with the predictions of the GUE model as suggested by Dyson [8]. An essential ingredient in the work of Rudnick–Sarnak is the following technical hypothesis involving the coefficients of $L(s,\pi)$. For $\operatorname{Re}(s)>1$, let us write

$$\frac{L'}{L}(s,\pi) = -\sum_{n=1}^{\infty}\frac{\Lambda_\pi(n)}{n^s},$$

where $\Lambda_\pi(n)$ is supported only on prime powers and is given by (2.6).

Hypothesis $\mathbf{H}_\pi$: Let $m\in\mathbb{N}$ and $\pi\in\mathscr{A}_m$. For any fixed $k\geqslant 2$,

$$\sum_{p \text{ prime}}\frac{|\Lambda_\pi(p^k)|^2}{p^k}<\infty.$$

The Ramanujan conjecture for cusp forms on GL_m suggests that $|\Lambda_\pi(p^k)| \leqslant m\log p$, which more than adequately satisfies the requirements for Hypothesis $\mathbf{H}_\pi$. Moreover, Hypothesis $\mathbf{H}_\pi$ is known for $m\leqslant 4$ due to Rudnick–Sarnak [47] and Kim [32]. It is also known in special cases when $m=5$ and $m=6$ due to Kim [31], Kim–Shahidi [33] and Wu–Ye [52]. Although the distribution of the coefficients of $L(s,\pi)$ is not universal, the theory of Rankin–Selberg L-functions, together with Hypothesis $\mathbf{H}_\pi$ ensures that the asymptotic relation

$$\sum_{n\leqslant X}\frac{|\Lambda_\pi(n)|^2}{n}\sim\frac{\log^2 X}{2} \tag{1.10}$$

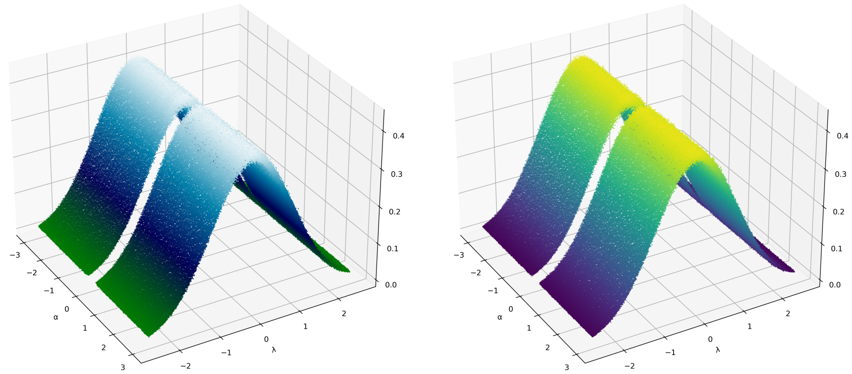

FIGURE 2. Left: A discrete approximation of the PCS using the zeros of $L(s,\Delta)$, the L-function associated with the modular discriminant, with $|\alpha| \geqslant 0.2$. Right: A discrete approximation of the PCS using the zeros of $L(s,E)$ where $E/\mathbb{Q}$ is the elliptic curve over $\mathbb{Q}$ defined by $E: y^2+y=x^3-x$, with $|\alpha| \geqslant 0.2$.

holds regardless of the choice of π. This shared behavior contributes to the universality of the n-level correlations with respect to π in [47]. In this context, we also mention the works of Katz–Sarnak [28, 29] and Iwaniec–Luo–Sarnak [23] on the distribution of low-lying zeros, that is, zeros close to the central point. Following their works, we believe that the distribution of the low-lying zeros is universal and predicted by only a few random matrix ensembles. For further insights on these topics, the reader is encouraged to consult the works of Heath-Brown [19], Keating–Snaith [30], Iwaniec–Sarnak [24], Mehta [38], Young [53] and the references there-in.

1.3. **A Pair Correlation Surface.** In addition to Figures 1–3, we present further numerical evidence in [1] related to the behavior of $S_\pi(X,T,\rho'_\pi)$. Based on these computations and graphical evidence, we propose the following conjectures on the statistical distribution of $S_\pi(X,T,\rho'_\pi)$. To proceed, we require some notation. Let $m \in \mathbb{N}$ and $\mathscr{A}_m$ be the set of irreducible cuspidal automorphic representations of GL_m over $\mathbb{Q}$ with unitary central character. For $\pi \in \mathscr{A}_m$, let $L(s,\pi)$ be its standard L-function. Let $X, T \geqslant 2$ and ρ'_π be a non-trivial zero of $L(s,\pi)$. Define

$$\mathcal{Q}_\pi(X,T,\rho'_\pi) := \frac{\operatorname{Re}\mathcal{S}_\pi(X,T,\rho'_\pi) - (m\log T)^{-1}(\min\{\log X, m\log T\})}{(1/2)(\min\{\log X, m\log T\})^{\frac{1}{2}}}, \tag{1.11}$$

where $\mathcal{S}_\pi(X,T,\rho'_\pi)$ is defined by (1.7).

Conjecture 1.1. *Let $m \in \mathbb{N}$ and $\pi \in \mathscr{A}_m$. For any $\alpha \in \mathbb{R}\setminus\{0\}$ and any $\lambda \in \mathbb{R}$,*

$$\lim_{T\to\infty} \frac{1}{N_\pi(T)} \#\left\{|\mathrm{Im}(\rho'_\pi)| \leqslant T\colon\, \mathcal{Q}_\pi(T^{|\alpha|m}, T, \rho'_\pi) < \lambda\right\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\lambda} e^{-\frac{1}{2}u^2}\,\mathrm{d}u, \tag{1.12}$$

where $N_\pi(T)$ is as in (1.5)*, ρ'_π runs over non-trivial zeros of $L(s,\pi)$ and $\mathcal{Q}_\pi(X,T,\rho'_\pi)$ is defined by* (1.11).

We now formulate the following definition of the Pair Correlation Surface (PCS) (if it exists). We define it pointwise as follows.

Definition 1.2. (Definition of the Pair Correlation Surface). *Let $m \in \mathbb{N}$ and $\pi \in \mathscr{A}_m$. Fix $\alpha \in \mathbb{R}\setminus\{0\}$, $\lambda \in \mathbb{R}$ and let*

$$g_\pi(\alpha,\lambda) := \lim_{\delta\to 0}\frac{1}{2\delta}\lim_{T\to\infty}\frac{1}{N_\pi(T)}\#\left\{|\mathrm{Im}(\rho'_\pi)| \leqslant T\colon\, \mathcal{Q}_\pi(T^{|\alpha|m}, T, \rho'_\pi) \in [\lambda-\delta, \lambda+\delta]\right\}, \tag{1.13}$$

provided that the limit exists. Let E_1 be the set of points $(\alpha,\lambda)\in(\mathbb{R}\setminus\{0\})\times\mathbb{R}$ for which $g_\pi(\alpha,\lambda)$ is defined. Let E_0 be the set of points $(0,\lambda)\in\{0\}\times\mathbb{R}$ for which the limit

$$g_\pi(0,\lambda):=\lim_{\substack{\alpha\to0\\ \alpha\neq0\\ (\alpha,\lambda)\in E_1}} g_\pi(\alpha,\lambda) \tag{1.14}$$

exists. Let $E=E_0\cup E_1$ and consider the function g defined on E by (1.13) *and* (1.14) *above. We call the graph of the function $g_\pi: E\to\mathbb{R}$ the Pair Correlation Surface associated to the zeros of $L(s,\pi)$.*

We make the following conjecture on the existence of the Pair Correlation Surface.

Conjecture 1.3. (Existence of the Pair Correlation Surface). *Let $m\in\mathbb{N}$ and $\pi\in\mathscr{A}_m$. For any $\alpha\in\mathbb{R}\setminus\{0\}$ and any $\lambda\in\mathbb{R}$, we have*

$$\lim_{\delta\to0}\frac{1}{2\delta}\lim_{T\to\infty}\frac{1}{N_\pi(T)}\#\Big\{|\mathrm{Im}(\rho'_\pi)|<T\colon \mathcal{Q}_\pi(T^{|\alpha|m},T,\rho'_\pi)\in[\lambda-\delta,\lambda+\delta]\Big\}=\frac{1}{\sqrt{2\pi}}e^{-\frac12\lambda^2}, \tag{1.15}$$

where $N_\pi(T)$ is given by (1.5)*, ρ'_π runs over non-trivial zeros of $L(s,\pi)$ and $\mathcal{Q}_\pi(X,T,\rho'_\pi)$ is defined by* (1.11)*. Moreover, the Pair Correlation Surface associated to the zeros of $L(s,\pi)$ is the graph of the function $g_\pi:\mathbb{R}^2\to\mathbb{R}$ defined by*

$$g_\pi(\alpha,\lambda)=\frac{1}{\sqrt{2\pi}}e^{-\frac12\lambda^2}. \tag{1.16}$$

We now present our main results. We focus on the case when $\alpha>0$. The case when $\alpha<0$ can be treated similarly. One can see that Conjecture 1.3 follows from Conjecture 1.1. In the direction of Conjecture 1.1, we prove the following result.

Theorem 1.4. *Let $m\in\mathbb{N}$ and $\pi\in\mathscr{A}_m$. Assume Hypothesis $\mathbf{H}_\pi$ and the Riemann Hypothesis for $L(s,\pi)$. Let $\{\alpha_j\}_{j\in\mathbb{N}}$ be any sequence of strictly positive real numbers decreasing to 0. Then there exists an increasing sequence $\{V_j\}_{j\in\mathbb{N}}$ tending to ∞ with the following property. For any sequence $\{T_j\}_{j\in\mathbb{N}}$ such that $T_j\geqslant V_j$ for all $j\in\mathbb{N}$, we have for any $\lambda\in\mathbb{R}$,*

$$\lim_{j\to\infty}\frac{1}{N_\pi(T_j)}\#\Big\{|\mathrm{Im}(\rho'_\pi)|\leqslant T_j\colon \mathcal{Q}_\pi(T_j^{\alpha_j m},T_j,\rho'_\pi)<\lambda\Big\}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\lambda}e^{-\frac12u^2}\,\mathrm{d}u, \tag{1.17}$$

where $N_\pi(T)$ is as in (1.5)*, ρ'_π runs over non-trivial zeros of $L(s,\pi)$ and $\mathcal{Q}_\pi(X,T,\rho'_\pi)$ is defined by* (1.11)*.*

In order to prove Theorem 1.4, we utilize the method of moments. For $r\in\mathbb{N}$, we compute the moments

$$\mathcal{M}_{\pi,r}(X,T):=\frac{1}{N_\pi(T)}\sum_{|\mathrm{Im}\,\rho'_\pi|\leqslant T}\left(\mathrm{Re}\,\mathcal{S}_\pi(X,T,\rho'_\pi)-\frac{\min\{\log X,m\log T\}}{m\log T}\right)^r. \tag{1.18}$$

These moments will furnish the distribution of $\mathcal{S}_\pi(X,T,\rho'_\pi)$. For $r\in\mathbb{N}$, let

$$\boldsymbol{\mu}_r:=\begin{cases}1\cdot3\cdots(r-1) & \text{if } r \text{ is even},\\ 0 & \text{if } r \text{ is odd}.\end{cases} \tag{1.19}$$

Theorem 1.5. *Let $m\in\mathbb{N},\pi\in\mathscr{A}_m$ and $\theta_m\in[0,\frac12-\frac{1}{m^2+1}]$ be an admissible exponent towards the Ramanujan conjecture for $L(s,\pi)$. Assume Hypothesis $\mathbf{H}_\pi$ and the Riemann Hypothesis for $L(s,\pi)$. Let $X,T\geqslant2, r\in\mathbb{N}$ and $\mathcal{M}_{\pi,r}(X,T)$ be as defined in* (1.18)*. Fix $\alpha\in\mathbb{R}$ such that*

$$0<\alpha<\frac{1}{mr(1+\frac43\theta_m)}.$$

If r is even, then

$$\mathcal{M}_{\pi,r}(T^{\alpha m},T)=\boldsymbol{\mu}_r\left(\frac{\alpha m\log T}{4}\right)^{\frac r2}+O_{\pi,r,\alpha}((\log T)^{\frac{r-1}{2}}),$$

where $\boldsymbol{\mu}_r$ *is given by* (1.19). *If* r *is odd, then*

$$\mathcal{M}_{\pi,r}(T^{\alpha m}, T) \ll_{\pi,r,\alpha} (\log T)^{\frac{r-1}{2}}.$$

We expect the results in Theorem 1.5 to hold for all fixed $\alpha \in \mathbb{R} \setminus \{0\}$. This leads us to make the following conjecture, which is evidently stronger than Conjectures 1.1 and 1.3.

Conjecture 1.6. *Let* $m \in \mathbb{N}$ *and* $\pi \in \mathscr{A}_m$. *Let* $X, T \geqslant 2, r \in \mathbb{N}$ *and* $\mathcal{M}_{\pi,r}(X,T)$ *be as defined in* (1.18). *Fix* $\alpha \in \mathbb{R} \setminus \{0\}$. *If* r *is even, then*

$$\mathcal{M}_{\pi,r}(T^{|\alpha|m}, T) = (\boldsymbol{\mu}_r + o(1)) \left(\frac{\min\{|\alpha|, 1\} m \log T}{4} \right)^{\frac{r}{2}}, \quad \text{as} \quad T \to \infty,$$

where $\boldsymbol{\mu}_r$ *is defined by* (1.19). *If* r *is odd, then*

$$\mathcal{M}_{\pi,r}(T^{|\alpha|m}, T) = o((\log T)^{\frac{r}{2}}) \quad \text{as} \quad T \to \infty.$$

We make the following remarks.

Remark 1.7. In some sense, Theorem 1.4 is as close as possible to Conjecture 1.1, while failing to prove it even for any single fixed α. We do need in Theorem 1.4 to let $\{\alpha_j\}_{j \in \mathbb{N}}$ tend to 0. But we can do so as slowly as we want.

Remark 1.8. The universality in the shape of the Pair Correlation Surface associated to $L(s,\pi)$ as seen from Theorem 1.4, while surprising, is also anticipated. When studying the pair correlation of zeros of L-functions, one typically uses an explicit formula to connect the zeros to primes, leading us to examine moments of sums of the form

$$\sum_{n \leqslant X} \frac{a_n \Lambda_\pi(n)}{n^{\rho_\pi}}, \quad \text{where} \quad \frac{L'}{L}(s,\pi) = \sum_{n=1}^{\infty} \frac{\Lambda_\pi(n)}{n^s},$$

and the sequence $\{a_n\}$ corresponds to Montgomery's weight function $w(u)$, or to more general weight functions. Importantly, if we write $\Lambda_\pi(p) = \lambda_\pi(p) \log p$ then the distribution of the coefficients $\lambda_\pi(p)$ is not universal across different π. If $\pi \in \mathscr{A}_2$, then there are two conjectured limiting distributions for the $\lambda_\pi(p)$'s: Sato–Tate or a uniform distribution with a Dirac mass term (see Serre [50]). As the degree increases, the number of possible limiting distributions increase. However, the Rankin-Selberg L-function theory shows that all these distributions share the same second moment under Hypothesis $\mathbf{H}_\pi$ (see Lemma 2.10). This common second moment is key to the universality in the n-level correlations, as shown by Rudnick and Sarnak [47] as well as in the shape of the Pair Correlation Surface.

Remark 1.9. The restriction for the range of α in Theorem 1.5 naturally arises in the calculation of the r-th moments and reflects upon Montgomery's result, in which case, the asymptotic (1.2) holds for $0 \leqslant \alpha < 1$. This restriction may be surpassed if one considers averaging over suitable families of L-functions. For instance, by averaging over certain families of Dirichlet L-functions and GL_2 L-functions, analogous versions of (1.2) have been established for extended ranges of α, see the works of Özlük [46], Chandee–Lee–Liu–Radziwiłł [5], and Chandee–Klinger-Logan–Li [4].

A significant observation is that Theorem 1.4 does not entirely address the uniformity in α with respect to T. Specifically, in (1.11) if X is small compared to T, say $X \asymp (\log T)^{\alpha m}$ for some fixed $0 < \alpha < 1/(2m)$ then as we shall see later, the asymptotic estimate for $\mathcal{S}_{\text{av},\pi}(X,T)$ is influenced by a term of order $X^{-2} \log T$. This phenomena mirrors the asymptotic relation (1.2) obtained by Montgomery for $\zeta(s)$. This sharp transition in the behavior of $\mathcal{S}_{\text{av},\pi}(X,T)$ accounts for the observed distortion in the distribution of $S_\pi(X,T,\rho'_\pi)$ for small ranges of X, as seen in Figure 1 and also in [1]. To explore this in more detail, we study a finer second-order distribution of $S_\pi(X,T,\rho'_\pi)$ when X is of size $(\log T)^\alpha$. We consider the shifted sums

$$\widetilde{\mathcal{S}}_\pi(X,T,\rho'_\pi) := \mathcal{S}_\pi(X,T,\rho'_\pi) - X^{-2}\frac{L'}{L}(\rho'_\pi - 2,\pi), \tag{1.20}$$

$$\operatorname{Re}\widetilde{\mathcal{S}}_{\mathrm{av},\pi}(X,T) := \frac{1}{N_\pi(T)}\sum_{|\operatorname{Im}\rho'_\pi|\leqslant T}\operatorname{Re}\widetilde{\mathcal{S}}_\pi(X,T,\rho'_\pi) \tag{1.21}$$

and define

$$\widetilde{\mathcal{Q}_\pi}(X,T,\rho'_\pi) := \frac{\operatorname{Re}\widetilde{\mathcal{S}_\pi}(X,T,\rho'_\pi) - (m\log T)^{-1}(\min\{\log X, m\log T\})}{(1/2)(\min\{\log X, m\log T\})^{\frac12}}, \tag{1.22}$$

where $\widetilde{\mathcal{S}_\pi}(X,T,\rho'_\pi)$ and $\widetilde{\mathcal{S}}_{\mathrm{av},\pi}(X,T)$ are defined by (1.20) and (1.21) respectively. For small ranges of X, we can relax the assumption of RH for $L(s,\pi)$ and instead adopt a weaker hypothesis regarding the density of zeros of $L(s,\pi)$ near the critical line.

Hypothesis $\mathbf{Z}_\pi$. Let $m\in\mathbb{N}$ and $\pi\in\mathscr{A}_m$. There exists a constant $A_\pi>0$ (depending on π) such that

$$N_\pi(\sigma,T) = |\{\rho_\pi = \beta_\pi + i\gamma_\pi : \sigma\leqslant\beta_\pi, |\gamma_\pi|\leqslant T, L(s,\rho_\pi)=0\}| \ll_\pi T^{1-A_\pi(\sigma-\frac12)}\log T,$$

uniformly for $\sigma\geqslant\frac12$ and $T\geqslant 2$.

Remark 1.10. Hypothesis $\mathbf{Z}_\pi$ is known for the Riemann zeta function and Dirichlet L-functions due to Selberg [48, 49] and for any $\pi\in\mathscr{A}_2$ due to Luo [35] and Beckwith–Liu–Thorner–Zaharescu [3].

Assuming Hypothesis $\mathbf{Z}_\pi$, our main result in this direction is as follows.

Theorem 1.11. *Let $m\in\mathbb{N}$ and $\pi\in\mathscr{A}_m$. Assume Hypothesis $\mathbf{H}_\pi$ and Hypothesis $\mathbf{Z}_\pi$ for $L(s,\pi)$. Let $\{\alpha_j\}_{j\in\mathbb{N}}$ be any sequence of strictly positive real numbers decreasing to 0. Then there exists an increasing sequence $\{V_j\}_{j\in\mathbb{N}}$ tending to ∞ with the following property. For any sequence $\{T_j\}_{j\in\mathbb{N}}$ such that $T_j\geqslant V_j$ for all $j\in\mathbb{N}$, we have for all $\lambda\in\mathbb{R}$,*

$$\lim_{j\to\infty}\frac{1}{N_\pi(T_j)}\#\left\{|\operatorname{Im}(\rho'_\pi)|\leqslant T_j\colon\ \widetilde{\mathcal{Q}}_\pi((\log T_j)^{\alpha_j m},T_j,\rho'_\pi)<\lambda\right\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\lambda} e^{-\frac12 u^2}\,\mathrm{d}u, \tag{1.23}$$

where $N_\pi(T)$ is as in (1.5)*, ρ'_π runs over non-trivial zeros of $L(s,\pi)$ and $\widetilde{\mathcal{Q}}_\pi(X,T,\rho'_\pi)$ is defined by* (1.22)*.*

To prove Theorem 1.11, we establish asymptotic estimates for the moments

$$\widetilde{\mathcal{M}}_{\pi,r}(X,T) := \frac{1}{N_\pi(T)}\sum_{|\operatorname{Im}\rho'_\pi|\leqslant T}\left(\operatorname{Re}\widetilde{\mathcal{S}}_\pi(X,T,\rho'_\pi) - \frac{\min\{\log X, m\log T\}}{m\log T}\right)^r. \tag{1.24}$$

Theorem 1.12. *Let $m\in\mathbb{N}$ and $\pi\in\mathscr{A}_m$. Assume Hypothesis $\mathbf{H}_\pi$ and Hypothesis $\mathbf{Z}_\pi$ for $L(s,\pi)$. Let $X,T\geqslant 3$, $r\in\mathbb{N}$ and $\widetilde{\mathcal{M}}_{\pi,r}(X,T)$ be as defined in* (1.24)*. Fix $\alpha\in\mathbb{R}$ such that $0<\alpha<4/(mr)$. If r is even, then*

$$\widetilde{\mathcal{M}}_{\pi,r}((\log T)^{\alpha m},T) = \boldsymbol{\mu}_r\left(\frac{\alpha m\log\log T}{4}\right)^{\frac r2} + O_{\pi,r,\alpha}((\log\log T)^{\frac{r-1}{2}}),$$

where $\boldsymbol{\mu}_r$ is given by (1.19)*. If r is odd, then*

$$\widetilde{\mathcal{M}}_{\pi,r}((\log T)^{\alpha m},T) \ll_{\pi,r,\alpha} (\log\log T)^{\frac{r-1}{2}}.$$

Remark 1.13. Theorems 1.11 and 1.12 hold unconditionally when $m=1,2$.

1.4. **Smooth Weight Functions and Distributions Associated to Triple Correlation.** Fix $X \geqslant 2$ and consider the function f on $(0, \infty)$ defined by

$$f(x) := \begin{cases} \left(\frac{x}{X}\right)^2, & x \leqslant X \\ \left(\frac{X}{x}\right)^2, & x \geqslant X. \end{cases} \tag{1.25}$$

Then (1.7) can be viewed as

$$\mathcal{S}_\pi(X, T, \rho'_\pi) = \sum_{|\mathrm{Im}\,\rho_\pi| \leqslant T} \hat{f}(\rho_\pi - \rho'_\pi), \tag{1.26}$$

where $\hat{f}$ denotes the Mellin transform of f. We first show that the results from Section 1.3 hold when f is replaced by a large class of smooth compactly supported weight functions.

Let $\Psi \in C_c^\infty(0, \infty)$ be a fixed non-negative compactly supported smooth function. Fix $X \geqslant 2$ and define

$$\Psi_X(x) := \Psi\left(\frac{x}{X}\right), \quad x \in \mathbb{R}. \tag{1.27}$$

Let $T \geqslant 2$ and define

$$\mathcal{S}_{\mathrm{av},\pi,\Psi}(X, T) := \frac{1}{N_\pi(T)} \sum_{|\mathrm{Im}\,\rho'_\pi| \leqslant T} \mathrm{Re}\,\mathcal{S}_{\pi,\Psi}(X, T, \rho'_\pi) \tag{1.28}$$

where

$$\mathcal{S}_{\pi,\Psi}(X, T, \rho'_\pi) := \sum_{|\mathrm{Im}\,\rho_\pi| \leqslant T} \hat{\Psi}_X(\rho_\pi - \rho'_\pi), \tag{1.29}$$

and ρ_π, ρ'_π vary over the non-trivial zeros of $L(s, \pi)$. For $r \in \mathbb{N}$, consider the moments

$$\mathcal{M}_{\pi,\Psi,r}(X, T) := \frac{1}{N_\pi(T)} \sum_{|\mathrm{Im}\,\rho'_\pi| \leqslant T} \left(\mathrm{Re}\,\mathcal{S}_{\pi,\Psi}(X, T, \rho'_\pi) - \frac{\min\{\log X, m\log T\}}{m \log T} \int_0^\infty \frac{\Psi^2(t)}{t}\,\mathrm{d}t\right)^r. \tag{1.30}$$

Our next result extends Theorem 1.5 for the above class of smooth compactly supported weight functions.

Theorem 1.14. *Let $m \in \mathbb{N}, \pi \in \mathscr{A}_m$ and $\theta_m \in [0, \frac{1}{2} - \frac{1}{m^2+1}]$ be an admissible exponent towards the Ramanujan conjecture for $L(s, \pi)$. Assume Hypothesis $\mathbf{H}_\pi$ and the Riemann Hypothesis for $L(s, \pi)$. Let $\Psi \in C_c^\infty(0, \infty)$ be a fixed non-negative compactly supported smooth function. Let $X, T \geqslant 2, r \in \mathbb{N}$ and $\mathcal{M}_{\pi,\Psi,r}(X, T)$ be as defined in* (1.30)*. Fix $\alpha \in \mathbb{R}$ such that*

$$0 < \alpha < \frac{1}{mr(1 + \theta_m)}.$$

If r is even, then

$$\mathcal{M}_{\pi,\Psi,r}(T^{\alpha m}, T) = \boldsymbol{\mu}_r \left(\frac{\alpha m \log T}{2} \int_0^\infty \frac{\Psi^2(t)}{t} dt\right)^{\frac{r}{2}} + O_{\pi,\Psi,r,\alpha}((\log T)^{\frac{r-1}{2}}),$$

where $\boldsymbol{\mu}_r$ is given by (1.19)*. If r is odd, then*

$$\mathcal{M}_{\pi,\Psi,r}(T^{\alpha m}, T) \ll_{\pi,\Psi,r,\alpha} (\log T)^{\frac{r-1}{2}}.$$

Similar to Conjecture 1.6, we expect the results in Theorem 1.14 to hold for all fixed $\alpha \in \mathbb{R} \setminus \{0\}$. Theorem 1.14 implies the following result on the distribution of $\mathcal{S}_{\pi,\Psi}(X, T, \rho'_\pi)$. Define

$$\mathcal{Q}_{\pi,\Psi}(X, T, \rho'_\pi) := \frac{\mathrm{Re}\,\mathcal{S}_{\pi,\Psi}(X, T, \rho'_\pi) - (m \log T)^{-1} \min\{\log X, m \log T\} \int_0^\infty t^{-1}\Psi^2(t)\,\mathrm{d}t}{(1/2)\left(\min\{\log X, m\log T\} \int_0^\infty t^{-1}\Psi^2(t)\,\mathrm{d}t\right)^{\frac{1}{2}}}, \tag{1.31}$$

where $\mathcal{S}_{\pi,\Psi}(X, T, \rho'_\pi)$ is defined by (1.26).

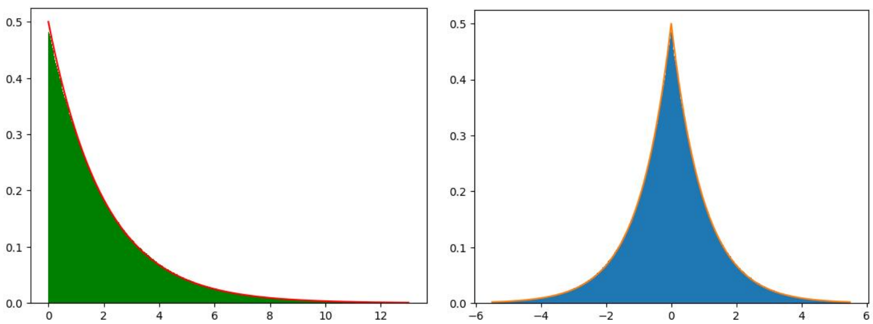

FIGURE 3. Left: A discrete approximation of the probability density function associated to $\widehat{\mathcal{S}}_{\pi,\Psi}(X_1, X_2, T, \rho_\pi)$ where $L(s,\pi) = \zeta(s)$, $X_1 = X_2 = T^{\frac{1}{2}}$ (in green). The probability distribution of a Chi-squared distribution with two degrees of freedom (in red). Right: A discrete approximation of the probability density function associated to $\widehat{\mathcal{S}}_{\pi,\Psi}(X_1, X_2, T, \rho_\pi)$ where $L(s,\pi) = \zeta(s)$, $X_1 = T^{\frac{1}{2}}$ and $X_2 = T^{\frac{2}{5}}$ (in blue). The probability distribution of a Laplace distribution with mean zero and scaling parameter one (in orange).

Theorem 1.15. *Let $m \in \mathbb{N}$ and $\pi \in \mathscr{A}_m$. Assume Hypothesis $\mathbf{H}_\pi$ and the Riemann Hypothesis for $L(s,\pi)$. Let $\Psi \in C_c^\infty(0,\infty)$ be a fixed non-negative compactly supported smooth function. Let $\{\alpha_j\}_{j\in\mathbb{N}}$ be any sequence of strictly positive real numbers decreasing to 0. Then there exists an increasing sequence $\{V_j\}_{j\in\mathbb{N}}$ tending to ∞ with the following property. For any sequence $\{T_j\}_{j\in\mathbb{N}}$ such that $T_j \geqslant V_j$ for all $j \in \mathbb{N}$, we have for any $\lambda \in \mathbb{R}$,*

$$\lim_{j\to\infty} \frac{1}{N_\pi(T_j)} \# \left\{ |\mathrm{Im}(\rho'_\pi)| \leqslant T_j \colon \mathcal{Q}_{\pi,\Psi}(T_j^{\alpha_j m}, T_j, \rho'_\pi) < \lambda \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\lambda} e^{-\frac{1}{2}u^2}\,\mathrm{d}u, \tag{1.32}$$

where $N_\pi(T)$ is as in (1.5)*, ρ'_π runs over non-trivial zeros of $L(s,\pi)$ and $\mathcal{Q}_{\pi,\Psi}(X,T,\rho'_\pi)$ is given by* (1.31).

Our final result concerns the distribution of non-trivial zeros of automorphic L-functions with respect to their triple correlation. Let $\Psi \in C_c^\infty(0,\infty)$ be a fixed non-negative compactly supported smooth function. Let $X_1, X_2, T \geqslant 2$ and consider the sums

$$\widehat{\mathcal{S}}_{\pi,\Psi}(X_1,X_2,T,\rho_\pi) := \sum_{|\mathrm{Im}\,\rho_{\pi,1}|\leqslant T} \sum_{|\mathrm{Im}\,\rho_{\pi,2}|\leqslant T} \hat{\Psi}_{X_1}(\rho_{\pi,1} - \rho_\pi)\hat{\Psi}_{X_2}(\rho_\pi - \rho_{\pi,2}), \tag{1.33}$$

where $\rho_{\pi,1}, \rho_{\pi,2}$ vary over the non-trivial zeros of $L(s,\pi)$. For $r \in \mathbb{N}$, consider the moments

$$\widehat{\mathcal{M}}_{\pi,\Psi,r}(X_1,X_2,T) := \frac{1}{N_\pi(T)} \sum_{|\mathrm{Im}\,\rho_\pi|\leqslant T} (\mathrm{Re}\,\widehat{\mathcal{S}}_{\pi,\Psi}(X_1,X_2,T,\rho_\pi))^r. \tag{1.34}$$

For $r \in \mathbb{N}$, let

$$\boldsymbol{\mathcal{L}}_r := \begin{cases} r! & \text{if } r \text{ is even,} \\ 0 & \text{if } r \text{ is odd,} \end{cases} \tag{1.35}$$

and

$$\boldsymbol{\chi}_r = 2^r \cdot r!. \tag{1.36}$$

Theorem 1.16. *Let $m \in \mathbb{N}, \pi \in \mathscr{A}_m$ and $\theta_m \in [0, \frac{1}{2} - \frac{1}{m^2+1}]$ be an admissible exponent towards the Ramanujan conjecture for $L(s,\pi)$. Assume Hypothesis $\mathbf{H}_\pi$ and the Riemann Hypothesis for $L(s,\pi)$. Let*

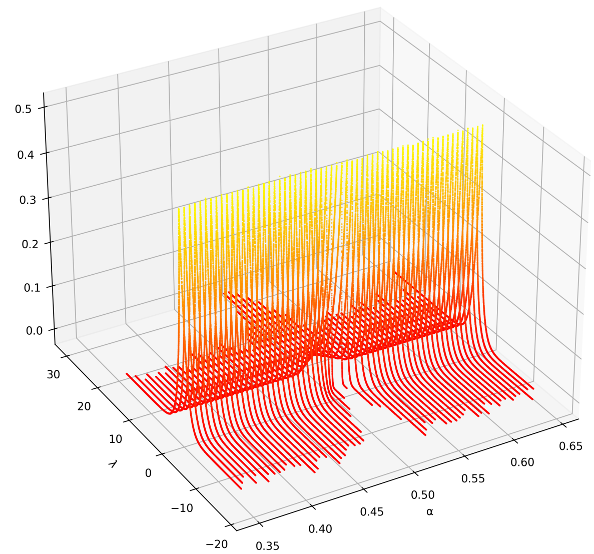

FIGURE 4. A discrete approximation of the probability density function associated to $\widehat{\mathcal{S}}_{\pi,\Psi}(X_1, X_2, T, \rho_\pi)$ where $L(s,\pi) = \zeta(s)$, $X_1 = T^{\frac{1}{2}}$ and $X_2 = T^{\alpha_2}$ with α_2 varying between $(0,1]$ in a discrete equi-spaced manner.

$\Psi \in C_c^\infty(0,\infty)$ be a fixed non-negative compactly supported smooth function. Let $X_1, X_2, T \geqslant 2, r \in \mathbb{N}$ and $\widehat{\mathcal{M}}_{\pi,\Psi,r}(X_1, X_2, T)$ be as defined in (1.34)*. Fix $\alpha_1, \alpha_2 > 0$ such that*

$$0 < \alpha_1 + \alpha_2 < \frac{1}{mr(1+\theta_m)}.$$

Then the following holds.

(1) *Suppose $\alpha_1 \neq \alpha_2$. If r is even, then*

$$\widehat{\mathcal{M}}_{\pi,\Psi,r}(T^{\alpha_1 m}, T^{\alpha_2 m}, T) = \boldsymbol{\mathcal{L}}_r \left(\frac{m\sqrt{\alpha_1\alpha_2}\log T}{2} \int_0^\infty \frac{\Psi^2(t)}{t}\,\mathrm{d}t \right)^r + O_{\pi,\Psi,r,\alpha_1,\alpha_2}((\log T)^{r-\frac{1}{2}}),$$

where $\boldsymbol{\mathcal{L}}_r$ is given by (1.35)*. If r is odd, then*

$$\widehat{\mathcal{M}}_{\pi,\Psi,r}(T^{\alpha_1 m}, T^{\alpha_2 m}, T) \ll_{\pi,\Psi,r,\alpha_1,\alpha_2} (\log T)^{\frac{r-1}{2}}.$$

(2) *Suppose $\alpha_1 = \alpha_2 = \alpha$. Then for any $r \in \mathbb{N}$,*

$$\widehat{\mathcal{M}}_{\pi,\Psi,r}(T^{\alpha m}, T^{\alpha m}, T) = \boldsymbol{\chi}_r \left(\frac{\alpha m \log T}{2} \int_0^\infty \frac{\Psi^2(t)}{t}\,\mathrm{d}t \right)^r + O_{\pi,\Psi,r,\alpha}((\log T)^{r-\frac{1}{2}}).$$

Remark 1.17. The moments in the case of triple correlation are *not* Gaussian. In fact, the elements of the sequence $\boldsymbol{\mathcal{L}}_r$, defined by (1.35), coincide with the moments of a random variable following the distribution Laplace(μ, b), where the mean μ is zero and the scaling parameter b is equal to 1. On the other hand, the elements of the sequence $\boldsymbol{\chi}_r$ defined by (1.36) are precisely the moments of a Chi-squared distribution χ_k^2 with degree of freedom k equal to 2. In particular, the distribution of $\operatorname{Re}\widehat{\mathcal{S}}_{\pi,\Psi}(T^{\alpha_1 m}, T^{\alpha_2 m}, T, \rho'_\pi)$ exhibits the characteristics of a Laplace distribution when $\alpha_1 \neq \alpha_2$ and that of a Chi-squared distribution when $\alpha_1 = \alpha_2$.

Remark 1.18. In Figure 4, we observe that the Laplace surface shifts as α_2 varies, until α_2 approaches α_1, at which point it begins transitioning into a χ_k^2 distribution. A natural question is what specific insights can be drawn about this phase transition, where the Laplace distribution evolves into the χ_k^2 distribution. In this context, our proof of Theorem 1.16 actually yields a slightly stronger result. Let $f : \mathbb{R} \to \mathbb{R}$ be any function such that $f(T) \to \infty$ as $T \to \infty$. For $T \geqslant 2$ sufficiently large, if

$$|\alpha_1 - \alpha_2| \geqslant \frac{f(T)}{\log T},$$

then we have Condition (1) from the statement of Theorem 1.16. Conversely, if

$$|\alpha_1 - \alpha_2| \leqslant \frac{1}{f(T)\log T},$$

then Condition (2) holds.

1.5. **Structure of the paper.** The paper is organized as follows. In Section 2, we provide background on automorphic L-functions and introduce preliminary lemmas which include explicit formulas for $L(s,\pi)$, Landau–Gonek type results, and some key asymptotic estimates related to Hypothesis $\mathbf{H}_\pi$. In Section 3, we carry out majority of our work towards estimating the moments in the case of pair correlation. Section 4 is devoted to the proofs of Theorems 1.4 and 1.5. In Section 5, we address the issue of uniformity in α with respect to T and prove results using Hypothesis $\mathbf{Z}_\pi$. The proofs of Theorems 1.11 and 1.12 are singled out in Section 6. In Section 7, we move on to a general class of smooth compactly supported weight functions. Finally, in Section 8 we study the distribution in the case of triple correlation and prove Theorem 1.16.

2. Preliminaries

2.1. **General Notations.** For most part of the paper, notation will be introduced when it is needed. Aside from that, we employ the following standard notations.

- Throughout the paper, the expressions $f(X) = O(g(X))$, $f(X) \ll g(X)$, and $g(X) \gg f(X)$ are equivalent to the statement that $|f(X)| \leqslant C|g(X)|$ for all sufficiently large X, where $C > 0$ is an absolute constant. A subscript of the form $\ll_\alpha$ means the implied constant may depend on the parameter α. Dependence on several parameters is indicated in an analogous manner, as in $\ll_{\alpha,\lambda}$.
- For any set $\mathcal{A}$, $\#\mathcal{A}$ denotes the cardinality of the set $\mathcal{A}$.
- For $s \in \mathbb{C}$, we denote the Mellin transform of $\varphi : (0,\infty) \to \mathbb{C}$ by
$$\hat{\varphi}(s) := \int_0^\infty \varphi(x) x^{s-1}\,\mathrm{d}x$$
when the integral exists.

2.2. **Background on L-functions.** We recall some standard facts about L-functions arising from automorphic representations. For more details, see Iwaniec–Kowalski [22], Jacquet [25], Jacquet–Shalika [27], Jacquet–Piateski-Shapiro–Shalika [26], Rudnick–Sarnak [47] and Shahidi [51].

Let $m \in \mathbb{N}$ and $\mathscr{A}_m$ be the set of irreducible cuspidal automorphic representations of GL_m over $\mathbb{Q}$ with unitary central character. Given $\pi \in \mathscr{A}_m$ with conductor q_π, let $L(s,\pi)$ be its standard L-function. There exist suitable complex numbers $\alpha_{1,\pi}(p), \alpha_{2,\pi}(p), \ldots, \alpha_{m,\pi}(p)$ such that

$$L(s,\pi) = \prod_{p \text{ prime}} \prod_{j=1}^{m} (1 - \alpha_{j,\pi}(p)p^{-s})^{-1} = \sum_{n=1}^{\infty} \frac{\lambda_\pi(n)}{n^s}. \tag{2.1}$$

Both the sum and the product in (2.1) converge absolutely for $\mathrm{Re}(s) > 1$. The $\alpha_{j,\pi}(p), 1 \leqslant j \leqslant m$ are called the local roots or local parameters of $L(s,\pi)$ at the prime p and they satisfy $|\alpha_{j,\pi}(p)| < p$ for all primes p. We also have spectral parameters $\kappa_\pi(1), \kappa_\pi(2), \ldots, \kappa_\pi(m) \in \mathbb{C}$ such that if we define

$$L(s,\pi_\infty) := \pi^{-\frac{ms}{2}} \prod_{j=1}^{m} \Gamma\left(\frac{s + \kappa_\pi(j)}{2}\right),$$

then the completed L-function

$$\Lambda(s,\pi) := (s(1-s))^{\delta_\pi} q_\pi^{s/2} L(s,\pi) L(s,\pi_\infty)$$

is entire of order one. Here we let $\delta_\pi = 1$ if $\pi \in \mathscr{A}_1$ is trivial; otherwise, $\delta_\pi = 0$. Let $\widetilde{\pi} \in \mathscr{A}_m$ be the contragredient representation. Then $\alpha_{j,\widetilde{\pi}}(p) = \overline{\alpha_{j,\pi}(p)}$ and $\kappa_{\widetilde{\pi}}(j) = \overline{\kappa_\pi(j)}$ for $j = 1, 2, \ldots, m$. Moreover, there exists a complex number ε_π of modulus one such that for all $s \in \mathbb{C}$,

$$\Lambda(s, \pi) = \varepsilon_\pi \Lambda(1 - s, \widetilde{\pi}). \tag{2.2}$$

The bounds

$$\log_p |\alpha_{j,\pi}(p)| \leqslant \theta_m, \quad \operatorname{Re}(\kappa_\pi(j)) \geqslant -\theta_m \tag{2.3}$$

hold for some

$$0 \leqslant \theta_m \leqslant \frac{1}{2} - \frac{1}{m^2 + 1}, \tag{2.4}$$

see Luo–Rudnick–Sarnak [36, 37] and Rudnick–Sarnak [47]. The Ramanujan conjecture and the Selberg eigenvalue conjecture assert that (2.3) hold with $\theta_m = 0$. This is not known except in specific cases, such as when $m = 1$, and for GL_2 L-functions corresponding to holomorphic forms due to Deligne [7]. For $\operatorname{Re}(s) > 1$, we take the logarithmic derivative of (2.1) to obtain

$$\frac{L'}{L}(s, \pi) = -\sum_{n=1}^{\infty} \frac{\Lambda_\pi(n)}{n^s}, \tag{2.5}$$

where $\Lambda_\pi(n)$ is supported on prime powers and is given by

$$\Lambda_\pi(p^k) = \sum_{j=1}^{m} \alpha_{j,\pi}(p)^k \log p. \tag{2.6}$$

Denote by $\rho_\pi = \beta_\pi + i\gamma_\pi$ the non-trivial zeros of $L(s, \pi)$. Define

$$N_\pi(\sigma, T) := \#\{\rho_\pi = \beta_\pi + i\gamma_\pi : \beta_\pi \geqslant \sigma, |\gamma_\pi| \leqslant T, L(\rho_\pi, \pi) = 0\}. \tag{2.7}$$

As with $\zeta(s)$, following a standard winding number argument, we can show that

$$N_\pi(T) := N_\pi(0, T) \sim \frac{T}{\pi} \log q_\pi T^m. \tag{2.8}$$

The Riemann Hypothesis (RH) for $L(s, \pi)$ asserts that $\operatorname{Re}(\rho_\pi) = \frac{1}{2}$.

2.3. **Sums over Zeros to Sums over Primes.** Landau [34, p. 353] showed that for $X > 1$ and not a prime power,

$$\sum_{n \leqslant X} \frac{\Lambda(n)}{n^s} = -\frac{\zeta'}{\zeta}(s) + \frac{X^{1-s}}{1-s} - \sum_\rho \frac{X^{\rho - s}}{\rho - s} + \sum_{n=1}^{\infty} \frac{X^{-2n-s}}{2n + s}, \tag{2.9}$$

where ρ runs over the non-trivial zeros of $\zeta(s)$ and $s \neq 1, s \neq \rho, s \neq -2n$. A truncated version of (2.9) was established by Gorodetsky [17]. The following lemma provides an analogous version of such results for L-functions arising from automorphic representations.

Lemma 2.1. *Let $m \in \mathbb{N}, \pi \in \mathscr{A}_m$ and $\theta_m \in [0, \frac{1}{2} - \frac{1}{m^2+1}]$ be an admissible exponent towards the Ramanujan conjecture for $L(s, \pi)$. Let $X \geqslant 2$ and*

$$T \geqslant 2 + |\operatorname{Im}(s)| + |\operatorname{Im}(\kappa_\pi(j))|,$$

for each $1 \leqslant j \leqslant m$ where $\kappa_\pi(j) \in \mathbb{C}$ are the spectral parameters. Let $s \in \mathbb{C}$ such that $L(s, \pi) \neq 0$. Then

$$\sum_{n \leqslant X} \frac{\Lambda_\pi(n)}{n^s} = -\frac{L'}{L}(s, \pi) + \delta_\pi \frac{X^{1-s}}{1-s} - \sum_{|\operatorname{Im}(\rho_\pi - s)| < T} \frac{X^{\rho_\pi - s}}{\rho_\pi - s} + \sum_{\substack{k \in \mathbb{N} \cup \{0\} \\ 1 \leqslant j \leqslant m}} \sum \frac{X^{-2k - \kappa_\pi(j) - s}}{2k + \kappa_\pi(j) + s} + E_\pi(X, T, s), \tag{2.10}$$

where $\delta_\pi = 1$ if $L(s,\pi) = \zeta(s)$ and zero otherwise,

$$E_\pi(X,T,s) \ll_\pi \frac{4^{|\mathrm{Re}(s)|} X^{\theta_m} \log X}{X^{\mathrm{Re}(s)}} \min\left(1, \frac{X}{T\langle X\rangle}\right) + \frac{4^{|\mathrm{Re}(s)|}\log^2(XT)}{T}\left(\frac{X^{1+\theta_m}}{X^{\mathrm{Re}(s)}} + \frac{1}{\log X}\right),$$

and $\langle X\rangle$ denotes the distance of X from the nearest prime power other than X itself. Furthermore, if X is not a prime power, we have the explicit formula

$$\sum_{n\leqslant X}\frac{\Lambda_\pi(n)}{n^s} = -\frac{L'}{L}(s,\pi) + \delta_\pi\frac{X^{1-s}}{1-s} - \sum_{\rho_\pi}\frac{X^{\rho_\pi - s}}{\rho_\pi - s} + \sum_{\substack{k\in\mathbb{N}\cup\{0\}\\ 1\leqslant j\leqslant m}}\frac{X^{-2k-\kappa_\pi(j)-s}}{2k+\kappa_\pi(j)+s}. \tag{2.11}$$

In (2.10) *and* (2.11)*, if $s=1$ and $\delta_\pi = 1$ then the term $X^{1-s}(1-s)^{-1} - L'/L(s,\pi)$ should be interpreted as $\log X - \gamma$, where γ is the Euler–Mascheroni constant.*

Proof. All implied constants are allowed to depend on π. Let $\sigma_0 = \max\{0, 1-\mathrm{Re}(s)\} + (\log X)^{-1}$. We apply Perron's formula to obtain

$$\sum_{n\leqslant X}\Lambda_\pi(n)n^{-s} = \frac{1}{2\pi i}\int_{\sigma_0 - i\infty}^{\sigma_0+i\infty} -\frac{L'}{L}(s+w,\pi)\frac{X^w}{w}\,\mathrm{d}w.$$

A standard argument [39, Corollary 5.3] allows us to truncate the above integral as follows:

$$\sum_{n\leqslant X}\frac{\Lambda_\pi(n)}{n^s} = \frac{1}{2\pi i}\int_{\sigma_0-iT}^{\sigma_0+iT} -\frac{L'}{L}(s+w,\pi)\frac{X^w}{w}\,\mathrm{d}w + R_\pi(X,T,s), \tag{2.12}$$

where $T \geqslant 2 + |\mathrm{Im}(s)| + |\mathrm{Im}(\kappa_\pi(j))|$ for each $1\leqslant j\leqslant m$ and

(2.13)

$$R_\pi(X,T,s) \ll_\pi \sum_{\substack{n=1\\ n\neq X}}^{\infty}\frac{\Lambda_\pi(n)}{n^{\mathrm{Re}(s)}}\left(\frac{X}{n}\right)^{\sigma_0}\min\left\{1, \frac{1}{T|\log(X/n)|}\right\} + \frac{4^{\sigma_0} + X^{\sigma_0}}{T}\left(-\frac{L'}{L}(\sigma_0 + \mathrm{Re}(s),\pi)\right).$$

When $n\leqslant 2X$ or $n\geqslant X/2$, $|\log(X/n)|$ has a positive lower bound and therefore, the contribution from these terms in (2.13) is

$$\ll_\pi \frac{X^{\sigma_0}}{T}\left(-\frac{L'}{L}(\sigma_0+\mathrm{Re}(s),\pi)\right) \ll_\pi \frac{X^{\max\{0,1-\mathrm{Re}(s)\}}\log X}{T}.$$

Consider next the terms for which $X/2 < n < X$. Let X_1 be the largest prime power less than X. We can assume $X/2 < X_1 < X$, since otherwise the terms under consideration vanish. For the term $n = X_1$, we have $\log(X/n) \geqslant (X-X_1)/X$ and therefore the contribution of this term to (2.13) is

$$\ll_\pi \frac{4^{|\mathrm{Re}(s)|}\Lambda_\pi(X_1)}{X^{\mathrm{Re}(s)}}\min\left(1, \frac{X}{T(X-X_1)}\right) \ll_\pi 4^{|\mathrm{Re}(s)|}X^{\theta_m - \mathrm{Re}(s)}(\log X)\min\left(1, \frac{X}{T(X-X_1)}\right).$$

For the other terms, we can put $n = X_1 - \nu$ where $0<\nu<X/2$ and then $\log(X/n)\geqslant \nu/X_1$. Hence the contribution of these terms to (2.13) is

$$\ll_\pi \frac{4^{|\mathrm{Re}(s)|}X^{1+\theta_m}\log X}{X^{\mathrm{Re}(s)}T}\sum_{0<\nu<X/2}\frac{1}{\nu} \ll_\pi \frac{4^{|\mathrm{Re}(s)|}X^{1+\theta_m-\mathrm{Re}(s)}\log^2 X}{T}.$$

A similar argument holds for the terms $X<n<2X$. Therefore combining all the cases, we obtain

$$\begin{aligned} R_\pi(X,T,s) &\ll_\pi \frac{4^{|\mathrm{Re}(s)|}X^{1+\theta_m-\mathrm{Re}(s)}\log^2 X}{T} + 4^{|\mathrm{Re}(s)|}X^{\theta_m-\mathrm{Re}(s)}(\log X)\min\left(1,\frac{X}{T\langle X\rangle}\right)\\ &\quad + \frac{4^{|\mathrm{Re}(s)|}X^{\max\{0,1-\mathrm{Re}(s)\}}\log X}{T}, \end{aligned} \tag{2.14}$$

where $\langle X\rangle$ denotes the distance of X from the nearest prime power other than X itself.

Our next step is to shift the path of integration in (2.12) to the left. Let K_π be a large positive number depending only on π satisfying $K_\pi > -\operatorname{Re}(s)$ that will be chosen later suitably. Since $T \geqslant 2 + |\operatorname{Im}(s)| + |\operatorname{Im}(\kappa_\pi(j))|$ for each $1 \leqslant j \leqslant m$, by [22, Proposition 5.7], there exists $T_1, T_2 \in [T, T+1]$ such that

$$\frac{L'}{L}\left(\sigma + i\operatorname{Im}(s) - iT_2, \pi\right), \frac{L'}{L}\left(\sigma + i\operatorname{Im}(s) + iT_1, \pi\right) \ll_\pi \log^2 T \tag{2.15}$$

uniformly for $-1 \leqslant \sigma \leqslant 2$. Note that the range of σ in [22, Proposition 5.7] can be extended to $-1 \leqslant \sigma \leqslant 2$ at the cost of amplifying the implied constant. Moreover, by extending the range of integration in (2.12) from $|\operatorname{Im}(w)| \leqslant T$ to $-T_2 \leqslant \operatorname{Im}(w) \leqslant T_1$, the additional error is at most

$$\ll_\pi \frac{X^{\sigma_0}}{T}\left(-\frac{L'}{L}(\sigma_0 + \operatorname{Re}(s), \pi)\right) \ll_\pi \frac{X^{\max\{0, 1-\operatorname{Re}(s)\}}\log X}{T},$$

which is acceptable. We replace the contour in (2.12) with a new contour $\mathcal{C}$ connecting the points $\sigma_0 - iT_2, -K_\pi - \operatorname{Re}(s) - iT_2, -K_\pi - \operatorname{Re}(s) + iT_1, \sigma_0 + iT_1$, in this order. By Cauchy's residue theorem, we get

$$\begin{aligned}\frac{1}{2\pi i}\int_{\sigma_0 - iT}^{\sigma_0 + iT} -\frac{L'}{L}(s+w,\pi)\frac{X^w}{w}\,\mathrm{d}w &= \frac{1}{2\pi i}\int_{\mathcal{C}} -\frac{L'}{L}(s+w,\pi)\frac{X^w}{w}\,\mathrm{d}w - \sum_{-T_2 < \operatorname{Im}(\rho_\pi - s) < T_1}\frac{X^{\rho_\pi - s}}{\rho_\pi - s}\\ &+ \sum_{\substack{k\in\mathbb{N}\cup\{0\}, 1\leqslant j\leqslant m\\ -K_\pi < \operatorname{Re}(-2k-\kappa_\pi(j))}}\frac{X^{-2k-\kappa_\pi(j)-s}}{2k+\kappa_\pi(j)+s} - \frac{L'}{L}(s,\pi) + \delta_\pi\frac{X^{1-s}}{1-s}.\end{aligned} \tag{2.16}$$

Here $\delta_\pi = 1$ if $L(s,\pi) = \zeta(s)$ and zero otherwise. Also, if $s = 1$ and $\delta_\pi = 1$, the integrand has a double pole at $w = 0$ and $X^{1-s}(1-s)^{-1} - \zeta'/\zeta(s)$ should be replaced with the residue $\log X - \gamma$ where γ is the Euler–Mascheroni constant. We may shorten the sum over $-T_2 < \operatorname{Im}(\rho - s) < T_1$ to a sum over $-T \leqslant \operatorname{Im}(\rho - s) \leqslant T$ with an additional error of at most

$$\ll_\pi \sum_{\operatorname{Im}(\rho_\pi - s)\in(T,T_1)\cup(-T_2,-T)}\frac{X^{1-\operatorname{Re}(s)}}{|\rho_\pi - s|} \ll_\pi \frac{X^{1-\operatorname{Re}(s)}\log T}{T}$$

which is again acceptable. Therefore, it remains to bound the integral over $\mathcal{C}$.

To bound the horizontal parts of our integral over $\mathcal{C}$, we break the range of integration $\operatorname{Re}(w) \in [-K_\pi - \operatorname{Re}(s), \sigma_0]$ into three separate parts. First consider $\operatorname{Re}(w) \in [-1 - \operatorname{Re}(s), \min\{2 - \operatorname{Re}(s), \sigma_0\}]$. We apply (2.15) to obtain

$$\frac{1}{2\pi i}\int_{-1-\operatorname{Re}(s)+iT_1}^{\min\{2-\operatorname{Re}(s),\sigma_0\}+iT_1} -\frac{L'}{L}(s+w,\pi)\frac{X^w}{w}\,\mathrm{d}w \ll_\pi \frac{\log^2 T}{T}\frac{X^{\min\{2-\operatorname{Re}(s),\sigma_0\}}}{\log X}, \tag{2.17}$$

and the same bound holds if T_1 is replaced with $-T_2$. Next, the contribution of $\operatorname{Re}(w) \in [2 - \operatorname{Re}(s), \sigma_0]$ should only be considered if this is a nonempty interval. In this case, we obtain

$$\frac{1}{2\pi i}\int_{2-\operatorname{Re}(s)+iT_1}^{\sigma_0+iT_1} -\frac{L'}{L}(s+w,\pi)\frac{X^w}{w}\,\mathrm{d}w \ll_\pi \frac{X^{\max\{0,1-\operatorname{Re}(s)\}}}{T\log X}. \tag{2.18}$$

The same bound holds if T_1 is replaced with $-T_2$. Now we consider the contribution from $\operatorname{Re}(w) \in [-K_\pi - \operatorname{Re}(s), -1 - \operatorname{Re}(s)]$. Let $D \subset \mathbb{C}$ be the half-plane $\{\sigma < -1 - \operatorname{Re}(s)\}$ minus a disc of radius $1/(4m)$ around each of the points $-2m - \kappa_\pi(j) - s$, where $m \in \mathbb{N}\cup\{0\}$ and $1 \leqslant j \leqslant m$. For any $w \in D$, by taking the log derivative of the functional equation (2.2) for $L(s,\pi)$ we have

$$\begin{aligned}-\frac{L'}{L}(s+w,\pi) &= \frac{L'}{L}(1-s-w,\widetilde{\pi})\\ &+ \frac{1}{2}\sum_{j=1}^{m}\left(\frac{\Gamma'}{\Gamma}\left(\frac{s+w+\kappa_\pi(j)}{2}\right) + \frac{\Gamma'}{\Gamma}\left(\frac{1-s-w+\kappa_{\widetilde{\pi}}(j)}{2}\right)\right) + O_\pi(1).\end{aligned} \tag{2.19}$$

Since $w \in D$, we have $L'/L(1-s-w,\widetilde{\pi}) \ll_\pi 1$. By Stirling's formula,

$$\sum_{j=1}^{m} \frac{\Gamma'}{\Gamma}\left(\frac{1-s-w+\kappa_{\widetilde{\pi}}(j)}{2}\right) \ll_\pi \log(|s+w|+2). \tag{2.20}$$

If $\mathrm{Re}(s+w+\kappa_\pi(j)) > \frac{1}{2}$, again by Stirling's formula,

$$\sum_{j=1}^{m} \frac{\Gamma'}{\Gamma}\left(\frac{s+w+\kappa_\pi(j)}{2}\right) \ll_\pi \log(|s+w|+2). \tag{2.21}$$

Otherwise, by using the reflection formula, we have

$$\sum_{j=1}^{m} \frac{\Gamma'}{\Gamma}\left(\frac{s+w+\kappa_\pi(j)}{2}\right) = \sum_{j=1}^{m} \frac{\Gamma'}{\Gamma}\left(1-\frac{s+w+\kappa_\pi(j)}{2}\right) - \sum_{j=1}^{m} \pi\cot\left(\pi\frac{s+w+\kappa_\pi(j)}{2}\right). \tag{2.22}$$

The sum involving Γ'/Γ on the right-hand side of (2.22) is again $\ll_\pi \log(|s+w|+2)$. Since $w \in D$, the cotangent sum in (2.22) is $O_\pi(1)$. Combining all the estimates, we see that for any $w \in D$,

$$\left|\frac{L'}{L}(s+w,\pi)\right| \ll_\pi \log(|s+w|+2). \tag{2.23}$$

Therefore, it follows that

$$\frac{1}{2\pi i}\int_{-K_\pi-\mathrm{Re}(s)+iT_1}^{-1-\mathrm{Re}(s)+iT_1} -\frac{L'}{L}(s+w,\pi)\frac{X^w}{w}\,\mathrm{d}w \ll_\pi \frac{\log T}{T}\int_{-K_\pi}^{-1} X^{a-\mathrm{Re}(s)}\,\mathrm{d}a \ll_\pi \frac{\log T}{T}\frac{X^{-1-\mathrm{Re}(s)}}{\log X}. \tag{2.24}$$

The same bound holds if T_1 is replaced with $-T_2$. The combined error from (2.17), (2.18) and (2.24) is acceptable. We may now choose K_π such that our contour lies inside D and therefore the integral over the vertical part of $\mathcal{C}$ is bounded using (2.23) again. To this end, we obtain

$$\begin{aligned}\frac{1}{2\pi i}\int_{-K_\pi-\mathrm{Re}(s)-iT_2}^{-K_\pi-\mathrm{Re}(s)+iT_1} -\frac{L'}{L}(s+w,\pi)\frac{X^w}{w}\,\mathrm{d}w &\ll_\pi \frac{\log(K_\pi T)}{K_\pi+\mathrm{Re}(s)}X^{-K_\pi-\mathrm{Re}(s)}\int_{-T_2}^{T_1}\mathrm{d}t\\ &\ll_\pi \frac{T\log(K_\pi T)X^{-K_\pi-\mathrm{Re}(s)}}{K_\pi+\mathrm{Re}(s)}.\end{aligned} \tag{2.25}$$

Letting K_π tend to ∞ while still ensuring the constraints on K_π, the bound in (2.25) tends to zero. Therefore combining all the error terms and putting together (2.12) and (2.16), we obtain (2.10).

If X is not a prime power, then by shifting X appropriately we may assume $\langle X\rangle \geqslant \frac{1}{2}$. Therefore letting T tending to ∞, $E_\pi(X,T,s)$ tends to zero and thus (2.11) follows. □

Let ρ_π and ρ'_π denote non-trivial zeros of $L(s,\pi)$. Recall from Section 1.1, for $X \geqslant 2$ and $T \geqslant 2$,

$$\mathcal{S}_\pi(X,T,\rho'_\pi) = \sum_{|\mathrm{Im}\,\rho_\pi|\leqslant T} X^{\rho_\pi-\rho'_\pi} w(\rho_\pi-\rho'_\pi),$$

$$\text{and}\quad \mathcal{S}_{\mathrm{av},\pi}(X,T) = \frac{1}{N_\pi(T)}\sum_{|\mathrm{Im}\,\rho'_\pi|\leqslant T} \mathrm{Re}\,\mathcal{S}_\pi(X,T,\rho'_\pi),$$

where $w(u)$ and $N_\pi(T)$ are given by (1.4) and (1.5). Fix $X \geqslant 2$. Define the sequence

$$a_n := \begin{cases} \left(\dfrac{n}{X}\right)^2, & n \leqslant X\\ \left(\dfrac{X}{n}\right)^2, & n \geqslant X. \end{cases} \tag{2.26}$$

Equipped with Lemma 2.1, we are able to establish the following result.

Lemma 2.2. *Let $m \in \mathbb{N}$ and $\pi \in \mathscr{A}_m$. Let $X, T \geqslant 2$ and $\{a_n\}$ be the sequence given by* (2.26)*. Then for all non-trivial zeros $\rho'_\pi = \beta'_\pi + i\gamma'_\pi$ of $L(s,\pi)$ with $|\gamma'_\pi| \leqslant T$ outside an exceptional set of size at most $O_\pi(\sqrt{T}\log T)$, we have*

$$\sum_{\rho_\pi} X^{\rho_\pi - \rho'_\pi} w(\rho_\pi - \rho'_\pi) = -\sum_{n=1}^{\infty} \frac{a_n \Lambda_\pi(n)}{n^{\rho'_\pi}} + \frac{m\log(|\gamma'_\pi|+3)}{X^2} + O_\pi\left(\frac{X^{1-\beta'_\pi}}{T} + \frac{1}{X^2}\right), \tag{2.27}$$

Proof. Fix a non-trivial zero $\rho'_\pi = \beta'_\pi + i\gamma'_\pi$ of $L(s,\pi)$. We may assume that $|\gamma'_\pi| > \sqrt{T}$. For X not a prime power, $s \neq 1$, and $L(s,\pi) \neq 0$, Lemma 2.1 shows that

$$\sum_{n \leqslant X} \frac{\Lambda_\pi(n)}{n^s} = -\frac{L'}{L}(s,\pi) + \delta_\pi \frac{X^{1-s}}{1-s} - \sum_{\rho_\pi} \frac{X^{\rho_\pi - s}}{\rho_\pi - s} + \sum_{\substack{k \in \mathbb{N}\cup\{0\} \\ 1 \leqslant j \leqslant m}}\sum \frac{X^{-2k-\kappa_\pi(j)-s}}{2k+\kappa_\pi(j)+s}, \tag{2.28}$$

where $\delta_\pi = 1$ if $L(s,\pi) = \zeta(s)$ and zero otherwise. Choosing $s = 2 + \rho'_\pi$ in (2.28) and using the Dirichlet series for $L'/L(s,\pi)$, we have

$$\sum_{\rho_\pi} \frac{X^{\rho_\pi - \rho'_\pi}}{-2 + (\rho_\pi - \rho'_\pi)} = \sum_{n > X} \frac{a_n \Lambda_\pi(n)}{n^{\rho'_\pi}} - \frac{X^{1-\rho'_\pi}}{1+\rho'_\pi} + \sum_{\substack{k \in \mathbb{N}\cup\{0\} \\ 1 \leqslant j \leqslant m}}\sum \frac{X^{-2k-\kappa_\pi(j)-\rho'_\pi}}{2k+2+\kappa_\pi(j)+\rho'_\pi}. \tag{2.29}$$

Similarly, choosing $s = -2 + \rho'_\pi$ in (2.28) yields

$$\sum_{\rho_\pi} \frac{X^{\rho_\pi - \rho'_\pi}}{2 + (\rho_\pi - \rho'_\pi)} = -\sum_{n \leqslant X} \frac{a_n \Lambda_\pi(n)}{n^{\rho'_\pi}} - X^{-2}\frac{L'}{L}(\rho'_\pi - 2, \pi) + \delta_\pi \frac{X^{1-\rho'_\pi}}{3-\rho'_\pi} + \sum_{\substack{k \in \mathbb{N}\cup\{0\} \\ 1 \leqslant j \leqslant m}}\sum \frac{X^{-2k-\kappa_\pi(j)-\rho'_\pi}}{2k-2+\kappa_\pi(j)+\rho'_\pi}. \tag{2.30}$$

Subtracting (2.29) from (2.30), we arrive at

$$\begin{aligned}\sum_{\rho_\pi} X^{\rho_\pi - \rho'_\pi} w(\rho_\pi - \rho'_\pi) &= -\sum_{n=1}^{\infty} \frac{a_n \Lambda_\pi(n)}{n^{\rho'_\pi}} - X^{-2}\frac{L'}{L}(\rho'_\pi - 2, \pi) + \frac{X^{1-\rho'_\pi}}{(3-\rho'_\pi)(1+\rho'_\pi)} \\ &\quad + \sum_{\substack{k \in \mathbb{N}\cup\{0\} \\ 1 \leqslant j \leqslant m}}\sum \frac{4X^{-2k-\kappa_\pi(j)-\rho'_\pi}}{(2k+\kappa_\pi(j)+\rho'_\pi)^2 - 4},\end{aligned} \tag{2.31}$$

Both sides of (2.31) are continuous when X is a prime power so we no longer exclude such cases. Note that for T sufficiently large depending on π, if $a \leqslant \sigma \leqslant b < 0$ in a fixed vertical strip $[a,b] \times \mathbb{R}$, then

$$-\frac{L'}{L}(\sigma + it, \pi) = m\log(|t|+3) + O_\pi(1), \quad |t| > \sqrt{T}, \tag{2.32}$$

uniformly in σ. This follows by taking the logarithmic derivative of the functional equation (2.2) and applying Stirling's formula. Substituting (2.32) into (2.31), we obtain

$$\sum_{\rho_\pi} X^{\rho_\pi - \rho'_\pi} w(\rho_\pi - \rho'_\pi) = -\sum_{n=1}^{\infty} \frac{a_n \Lambda_\pi(n)}{n^{\rho'_\pi}} + \frac{m\log(|\gamma'_\pi|+3)}{X^2} + O_\pi\left(\frac{X^{1-\beta'_\pi}}{T} + \frac{1}{X^2}\right),$$

which completes the proof. ☐

Remark 2.3. The secondary main term $mX^{-2}\log(|\gamma'_\pi|+3)$ in (2.27) is the key reason behind the distortion in the shape of the Pair Correlation Surface when X is small compared to T. We address this situation in more detail in Sections 5 and 6.

Next, we replace $\mathcal{S}_\pi(X, T, \rho'_\pi)$ by a sum over prime powers.

Lemma 2.4. *Let $m \in \mathbb{N}$ and $\pi \in \mathscr{A}_m$. Let $X, T \geqslant 2$ and $\{a_n\}$ be the sequence given by (2.26). Then for all non-trivial zeros $\rho'_\pi = \beta'_\pi + i\gamma'_\pi$ of $L(s,\pi)$ with $|\gamma'_\pi| \leqslant T$ outside an exceptional set of size at most $O_\pi(\sqrt{T}\log T)$, we have*

$$\mathcal{S}_\pi(X,T,\rho'_\pi) = -\sum_{n=1}^{\infty}\frac{a_n\Lambda_\pi(n)}{n^{\rho'_\pi}} + \frac{m\log(|\gamma'_\pi|+3)}{X^2} + O_\pi\Big(\frac{X^{1-\beta'_\pi}}{T} + \frac{1}{X^2} + \frac{X\log T}{\sqrt{T}}\Big).$$

Moreover, assuming the Riemann Hypothesis for $L(s,\pi)$, we have

$$\mathcal{S}_\pi(X,T,\rho'_\pi) = -\sum_{n=1}^{\infty}\frac{a_n\Lambda_\pi(n)}{n^{\rho'_\pi}} + \frac{m\log(|\gamma'_\pi|+3)}{X^2} + O_\pi\Big(\frac{\sqrt{X}}{T} + \frac{1}{X^2} + \frac{\log T}{\sqrt{T}}\Big).$$

Proof. All implied constants are allowed to depend on π. Except for $O_\pi(\sqrt{T}\log T)$ non-trivial zeros $\rho'_\pi = \beta'_\pi + i\gamma'_\pi$ of $L(s,\pi)$, we may assume $\sqrt{T} \leqslant |\gamma'_\pi| \leqslant T - \sqrt{T}$. For such ρ'_π, we write

$$\sum_{\rho_\pi} X^{\rho_\pi-\rho'_\pi} w(\rho_\pi-\rho'_\pi) - \mathcal{S}_\pi(X,T,\rho'_\pi)$$

$$= \sum_{m=0}^{\infty}\sum_{\gamma_\pi - T\in(m,m+1]}\frac{X^{\rho_\pi-\rho'_\pi}}{4+(\gamma_\pi-\gamma'_\pi)^2} + \sum_{m=0}^{\infty}\sum_{\gamma_\pi + T\in(-m,-m-1]}\frac{X^{\rho_\pi-\rho'_\pi}}{4+(\gamma_\pi-\gamma'_\pi)^2}. \tag{2.33}$$

We focus on the first sum on the right hand side of (2.33), i.e. the case when $\gamma_\pi > T$. By estimates on the local density of zeros of $L(s,\pi)$(see [22, Prop. 5.7]), the first sum is

$$\ll_\pi \frac{X\log(T+1)}{4+(T-\gamma')^2} + \sum_{m=1}^{\infty}\sum_{\gamma_\pi - T\in(m,m+1]}\frac{X^{\rho_\pi-\rho'_\pi}}{4+(\gamma_\pi-\gamma'_\pi)^2}$$

$$\ll_\pi \frac{X\log T}{T} + \int_{T+1}^{\infty}\frac{X\log u}{(u-\gamma')^2}\,\mathrm{d}u \ll_\pi \frac{X\log T}{\sqrt{T}}. \tag{2.34}$$

A similar argument holds for the second sum on the right hand side of (2.33), i.e. the case when $\gamma_\pi < -T$. Moreover, assuming RH for $L(s,\pi)$, we have $|X^{\rho_\pi-\rho'_\pi}| \leqslant 1$ and the bound for (2.34) is $\ll_\pi T^{-1/2}\log T$. Now applying Lemma 2.2, the desired conclusion follows. □

2.4. **Landau–Gonek Generalizations for $L(s,\pi)$.** We begin with a refined form of Gonek's result [16] on sums over zeros of L-functions due to Murty–Zaharescu [43] and Ford–Soundararajan–Zaharescu [9].

Lemma 2.5. *Let $m \in \mathbb{N}, \pi \in \mathscr{A}_m$ and $\theta_m \in [0, \frac{1}{2} - \frac{1}{m^2+1}]$ be an admissible exponent towards the Ramanujan conjecture for $L(s,\pi)$. Let $x > 1, T \geqslant 2$ and denote by n_x the nearest prime power to x. Then for any $\varepsilon > 0$*

$$\sum_{|\gamma_\pi|\leqslant T} x^{\rho_\pi} = -\frac{\Lambda_\pi(n_x)}{\pi}\frac{\sin(T\log(x/n_x))}{\log(x/n_x)} + O_{\pi,\varepsilon}\Big(x^{1+\theta_m+\varepsilon}\log T + \frac{\log T}{\log x}\Big), \tag{2.35}$$

where if $x = n_x$, the first term on the right hand side of (2.35) is $-\frac{T\Lambda_\pi(x)}{\pi}$.

Proof. The main term is obtained from [43, Proposition 1] and [10, Lemma 1]. To bound the error terms, we follow [9, Lemma 2]. In their notation, we let $F = L(s,\pi)$. The desired conclusion follows immediately. □

A quick consequence of the above lemma are the following two corollaries.

Corollary 2.6. *Let $m \in \mathbb{N}, \pi \in \mathscr{A}_m$ and $\theta_m \in [0, \frac{1}{2} - \frac{1}{m^2+1}]$ be an admissible exponent towards the Ramanujan conjecture for $L(s,\pi)$. Let $T \geqslant 2$ and suppose $a, b \in \mathbb{N}$ with $1 \leqslant b < a \leqslant T^\xi$ for some $\xi > 0$. Then for any $\varepsilon > 0$*

$$\sum_{|\gamma_\pi|\leqslant T}\Big(\frac{a}{b}\Big)^{\rho_\pi} = -\frac{T}{\pi}\Lambda_\pi\Big(\frac{a}{b}\Big) + O_{\pi,\xi,\varepsilon}(a^{1+\theta_m+\varepsilon}\log^2 T). \tag{2.36}$$

Moreover, assuming the Riemann Hypothesis for $L(s,\pi)$, for any $\varepsilon>0$

$$\sum_{|\gamma_\pi|\leqslant T}\left(\frac{a}{b}\right)^{i\gamma_\pi}=-\frac{T}{\pi}\sqrt{\frac{b}{a}}\Lambda_\pi\left(\frac{a}{b}\right)+O_{\pi,\xi,\varepsilon}((ab)^{\frac12+\theta_m+\varepsilon}\log^2 T). \tag{2.37}$$

Proof. First assume that a/b is not a prime power. Choose $x=a/b$ in (2.35). Then $x\neq n_x$ and $|\log(x/n_x)|\gg a^{-1}$. Therefore the first term on the right hand side of (2.35) is $\ll_{\pi,\xi} a\log T$. The error term in (2.35) in this case is

$$\ll_{\pi,\xi,\varepsilon}\left(\frac{a}{b}\right)^{1+\theta_m+\varepsilon}\log^2 T+a\log T,$$

for any $\varepsilon>0$ since $\log x\geqslant a^{-1}$. Therefore, we obtain

$$\sum_{|\gamma_\pi|\leqslant T}\left(\frac{a}{b}\right)^{\rho_\pi}\ll_{\pi,\xi,\varepsilon} a^{1+\theta_m+\varepsilon}\log^2 T.$$

When a/b is a prime power, the main term in (2.35) is $-\frac{T\Lambda_\pi(a/b)}{\pi}$ and the proof of (2.36) is complete. Assuming RH for $L(s,\pi)$, we have $\rho_\pi=\frac12+i\gamma_\pi$ and thus (2.37) follows from (2.36). □

Corollary 2.7. *Let $m\in\mathbb{N},\pi\in\mathscr{A}_m$ and $\theta_m\in[0,\frac12-\frac{1}{m^2+1}]$ be an admissible exponent towards the Ramanujan conjecture for $L(s,\pi)$. Let $T\geqslant 2$ and suppose $a,b\in\mathbb{N}$ with $1\leqslant a<b\leqslant T^\xi$ for some $\xi>0$. Then for any $\varepsilon>0$*

$$\sum_{|\gamma_\pi|\leqslant T}\left(\frac{a}{b}\right)^{\rho_\pi}=-\frac{T}{\pi}\frac{a}{b}\Lambda_{\widetilde{\pi}}\left(\frac{b}{a}\right)+O_{\pi,\xi,\varepsilon}(ab^{\theta_m+\varepsilon}\log^2 T). \tag{2.38}$$

Moreover, assuming the Riemann Hypothesis for $L(s,\pi)$, for any $\varepsilon>0$

$$\sum_{|\gamma_\pi|\leqslant T}\left(\frac{a}{b}\right)^{i\gamma_\pi}=-\frac{T}{\pi}\sqrt{\frac{a}{b}}\Lambda_{\widetilde{\pi}}\left(\frac{b}{a}\right)+O_{\pi,\xi,\varepsilon}((ab)^{\frac12+\theta_m+\varepsilon}\log^2 T). \tag{2.39}$$

Proof. Note that

$$\sum_{|\gamma_\pi|\leqslant T}\left(\frac{a}{b}\right)^{\rho_\pi}=\frac{a}{b}\sum_{|\gamma_\pi|\leqslant T}\left(\frac{b}{a}\right)^{1-\rho_\pi}=\frac{a}{b}\sum_{|\gamma_{\widetilde{\pi}}|\leqslant T}\left(\frac{b}{a}\right)^{\rho_{\widetilde{\pi}}}.$$

Applying Corollary 2.6, we obtain

$$\sum_{|\gamma_\pi|\leqslant T}\left(\frac{a}{b}\right)^{\rho_\pi}=-\frac{T}{\pi}\frac{a}{b}\Lambda_{\widetilde{\pi}}\left(\frac{b}{a}\right)+O_{\pi,\xi,\varepsilon}(ab^{\theta_m+\varepsilon}\log^2 T). \tag{2.40}$$

Assuming RH for $L(s,\pi)$, we have $\rho_\pi=\frac12+i\gamma_\pi$ and thus (2.39) follows from (2.40). □

2.5. Some Key Lemmas and Estimates.

We record the following result for later use.

Lemma 2.8. *Let $z\in\mathbb{C}$ and $k\in\mathbb{N}$. Then*

$$\mathrm{Re}(z)^{2k}=\frac{1}{2^{2k}}\binom{2k}{k}\mathrm{Re}(z^k\overline{z}^k)+\frac{1}{2^{2k-1}}\sum_{i=0}^{k-1}\binom{2k}{i}\mathrm{Re}(z^{2k-i}\overline{z}^i),$$

$$\text{and}\quad \mathrm{Re}(z)^{2k+1}=\frac{1}{4^k}\sum_{i=0}^{k}\binom{2k+1}{i}\mathrm{Re}(z^{2k+1-i}\overline{z}^i).$$

Proof. The proof is a direct application of the following two trigonometric identities:

$$\cos^{2k}\theta = \frac{1}{2^{2k}}\binom{2k}{k} + \frac{1}{2^{2k-1}}\sum_{i=0}^{k-1}\binom{2k}{i}\cos[2(k-i)\theta],$$

$$\text{and} \quad \cos^{2k+1}\theta = \frac{1}{4^k}\sum_{i=0}^{k}\binom{2k+1}{i}\cos[(2k+1-2i)\theta].$$

□

It is essential for later analysis to understand variants of the following sum

$$\sum_{n\leqslant X}\frac{|\Lambda_\pi(n)|^2}{n}. \tag{2.41}$$

To do so, we require the following hypothesis of Rudnick–Sarnak [47].

Hypothesis $\mathbf{H}_\pi$: Let $m \in \mathbb{N}$ and $\pi \in \mathscr{A}_m$. For any fixed $k \geqslant 2$,

$$\sum_{p \text{ prime}}\frac{|\Lambda_\pi(p^k)|^2}{p^k} < \infty. \tag{2.42}$$

Lemma 2.9. *Let $m \in \mathbb{N}$ and $\pi \in \mathscr{A}_m$. Let $X \geqslant 2$. If π satisfies Hypothesis $\mathbf{H}_\pi$ then*

$$\sum_{n\leqslant X}\frac{|\Lambda_\pi(n)|^2}{n} = \frac{\log^2 X}{2} + O_\pi(1).$$

Proof. As in [47, Eq. (2.25)], we use the upper bound for θ_m and Hypothesis $\mathbf{H}_\pi$ to show that the contribution from the prime powers is $O_\pi(1)$. Hence, we are just concerned with the sum over primes. When π is self-contragredient, this follows from [13, Theorem 1.1]. Due to the zero-free region of Humphries and Thorner [21, Theorem 2.1], the self-contragredient condition is no longer necessary. □

Lemma 2.10. *Let $m \in \mathbb{N}, \pi \in \mathscr{A}_m$ and assume Hypothesis $\mathbf{H}_\pi$. Let $2 \leqslant X \leqslant V$ and suppose $\{a_n\}$ be the sequence given by* (2.26). *Then*

$$\sum_{n\leqslant V}\frac{a_n|\Lambda_\pi(n)|^2}{n} = \log X - \frac{\log V}{2}\left(\frac{X}{V}\right)^2 + O_\pi(1),$$

$$\textit{and} \quad \sum_{n\leqslant V}\frac{a_n^2|\Lambda_\pi(n)|^2}{n} = \frac{\log X}{2} - \frac{\log V}{4}\left(\frac{X}{V}\right)^4 + O_\pi(1).$$

Proof. The proof follows by partial summation and Lemma 2.9. □

Lemma 2.11. *Let $m \in \mathbb{N}$ and $\pi \in \mathscr{A}_m$. For any $j \in \mathbb{N}$ with $j > 1$, we have*

$$\sum_{p\leqslant X}\frac{|\Lambda_\pi(p)|^{2j}}{p^j} < \infty.$$

Proof. We adapt and modify the arguments from [47, Page 15 and Prop. 2.3], and for clarity, we temporarily adopt the notations used in [47]. By (2.3), it is sufficient to focus on the case where $j = 2$. We work with the L-function $L(s, (\pi\times\pi)\times(\widetilde{\pi}\times\widetilde{\pi}))$. Consider the logarithmic derivative:

$$\frac{L_S'}{L_S}(s, (\pi\times\pi)\times(\widetilde{\pi}\times\widetilde{\pi})) = -\sum_{(n,S)=1}\frac{\Lambda(n)\,|a_\pi(n)|^4}{n^s} \tag{2.43}$$

where S is the set of ramified primes and $a_\pi(n)$ is given by [47, Eq. 2.13]. The contribution from the ramified primes and their powers is at most $O(1)$. Differentiating (2.43) three times, we define:

$$G(s) := \left(\frac{L_S'}{L_S}\right)^{(3)}(s+2, (\pi\times\pi)\times(\widetilde{\pi}\times\widetilde{\pi})) = \sum_{(n,S_\pi)=1} \frac{(\log n)^3\Lambda(n)\,|a_\pi(n)|^4}{n^{s+2}}. \tag{2.44}$$

Notice that $L(s,(\pi\times\pi)\times(\widetilde{\pi}\times\widetilde{\pi}))$ has a simple pole at $s=1$, which implies that $G(s)$ has a pole of order four at $s=-1$. From here, the argument proceeds similarly to that in [47], with two notable changes. First, the residue of $G(s)x^s(s(s+1))^{-1}$ at $s=-1$ is bounded by $\ll X^{-1}(\log X)^4$. Second, the sum over the zeros can be bounded using the zero-free region from Humphries and Thorner [21, Theorem 2.1]. □

3. Moments of Sums over Primes: Part I

By Lemma 2.4, it suffices to focus on sums over primes to study the distribution of $\mathcal{S}_\pi(X,T,\rho_\pi')$. Fix $\xi>1$. Suppose $T\geqslant 2$ and let α, X, V satisfy

$$0<\alpha<\frac{1}{m}, \quad X=T^{\alpha m} \quad \text{and} \quad V=X^\xi. \tag{3.1}$$

We begin by estimating certain mixed moments of sums over primes.

Lemma 3.1. *Let $m\in\mathbb{N}, \pi\in\mathscr{A}_m$ and $\theta_m\in[0,\frac12-\frac{1}{m^2+1}]$ be an admissible exponent towards the Ramanujan conjecture for $L(s,\pi)$. Assume Hypothesis $\mathbf{H}_\pi$ and the Riemann Hypothesis for $L(s,\pi)$. Let k,ℓ be non-negative integers with $k+\ell=2d$ for some $d\in\mathbb{N}$. Let $T\geqslant 2$ and α,ξ,X,V be as in (3.1) satisfying*

$$\alpha<\frac{1}{2dm(1+\xi\theta_m)}.$$

Let $\{a_n\}$ be the sequence defined by (2.26). Then

$$\frac{1}{N_\pi(T)}\sum_{|\gamma_\pi|\leqslant T}\left(\sum_{p\leqslant V}\frac{a_p\Lambda_\pi(p)}{p^{\frac12+i\gamma_\pi}}\right)^k\left(\sum_{q\leqslant V}\frac{a_q\overline{\Lambda_\pi(q)}}{q^{\frac12-i\gamma_\pi}}\right)^\ell = \boldsymbol{\delta}_{k\ell}d!\left(\sum_{p\leqslant V}\frac{a_p^2|\Lambda_\pi(p)|^2}{p}\right)^d + O_{\pi,d,\alpha,\xi}(\max\{1,(\log T)^{d-2}\}).$$

Here p,q run over primes up to V and $\boldsymbol{\delta}_{k\ell}=1$ if $k=\ell$, and zero otherwise.

Proof. All implied constants in the proof may depend at most on π,d,α and ξ. For notational simplicity, let

$$\mathcal{A}_{k,\ell} := \frac{1}{N_\pi(T)}\sum_{|\gamma_\pi|\leqslant T}\left(\sum_{p\leqslant V}\frac{a_p\Lambda_\pi(p)}{p^{\frac12+i\gamma_\pi}}\right)^k\left(\sum_{q\leqslant V}\frac{a_q\overline{\Lambda_\pi(q)}}{q^{\frac12-i\gamma_\pi}}\right)^\ell. \tag{3.2}$$

and $\eta:=\eta(\pi,d,\alpha,\xi)=2\alpha dm(1+\xi\theta_m)$. Note that $\eta<1$. We break our proof into two cases.

Case 1 : When $k\neq\ell$. Assume $k>\ell$. Consider the situation when $\ell=0$. Then

$$\mathcal{A}_{k,0}=\frac{1}{N_\pi(T)}\sum_{p_1,p_2,\ldots,p_k\leqslant V}\left(\prod_{i=1}^k a_{p_i}\Lambda_\pi(p_i)\right)\sum_{|\gamma_\pi|\leqslant T}P^{-\frac12-i\gamma_\pi},$$

where $P=p_1p_2\cdots p_k$. By Corollary 2.7,

$$\sum_{|\gamma_\pi|\leqslant T}P^{-i\gamma_\pi}=-\frac{T\Lambda_\pi(P)}{\pi\sqrt{P}}+O_{\pi,\varepsilon}(P^{\frac12+\theta_m+\varepsilon}\log^2 T).$$

for any $\varepsilon>0$. Note $\Lambda_\pi(P)=0$ for those tuples $(p_1,p_2,\ldots,p_k)$ where all p_i's are not equal. Hence we get

$$\mathcal{A}_{k,0}=-\frac{T}{\pi N_\pi(T)}\sum_{p\leqslant V}\frac{a_p^k\Lambda_\pi^k(p)\Lambda_\pi(p^k)}{p^k}+O_{\pi,d,\xi}\left(\frac{V^{2d\theta_m+\varepsilon}\log T}{T}\sum_{p_1,\ldots,p_k\leqslant V}\prod_{i=1}^k a_{p_i}|\Lambda_\pi(p_i)|\right). \tag{3.3}$$

By Cauchy–Schwarz and Lemma 2.10, the error term in (3.3) is

$$\ll_{\pi,d,\alpha,\xi,\varepsilon} \frac{V^{2d\theta_m+\varepsilon}\log T}{T}\bigg(\sum_{p\leqslant V} pa_p\bigg)^d\bigg(\sum_{p\leqslant V}\frac{a_p|\Lambda_\pi(p)|^2}{p}\bigg)^d \ll_{\pi,d,\alpha,\xi,\varepsilon} T^{\eta-1+\varepsilon}.$$

Since $k\geqslant 2$, by Cauchy–Schwarz, Hypothesis $\mathbf{H}_\pi$ and Lemma 2.11, the first term on the right hand side of (3.3) is $\ll_{\pi,d,\alpha,\xi} (\log T)^{-1}$. Hence by choosing $\varepsilon>0$ sufficiently small, we obtain

$$\mathcal{A}_{k,0} \ll_{\pi,d,\alpha,\xi} (\log T)^{-1}. \tag{3.4}$$

Now, assume $\ell>0$. Then

$$\mathcal{A}_{k,\ell} = \frac{1}{N_\pi(T)}\sum_{\substack{p_1,p_2,\ldots,p_k\leqslant V\\ q_1,q_2,\ldots,q_\ell\leqslant V}}\frac{(\prod_{i=1}^k a_{p_i}\Lambda_\pi(p_i))(\prod_{j=1}^\ell a_{q_j}\overline{\Lambda_\pi(q_j)})}{(PQ)^{\frac12}}\sum_{|\gamma_\pi|\leqslant T}\bigg(\frac{Q}{P}\bigg)^{i\gamma_\pi},$$

where $P=p_1p_2\cdots p_k$ and $Q=q_1q_2\cdots q_\ell$. Note that $P\neq Q$. We break the sum $\mathcal{A}_{k,\ell}$ by writing

$$\mathcal{A}_{k,\ell} = \mathcal{A}_{k,\ell}(P>Q)+\mathcal{A}_{k,\ell}(P<Q),$$

depending on whether $P>Q$ or $P<Q$. Since $Q/P\notin\mathbb{Z}$, using Corollary 2.6, we have

$$\mathcal{A}_{k,\ell}(P<Q) \ll_{\pi,d,\alpha,\xi,\varepsilon} T^{\eta-1+\varepsilon}. \tag{3.5}$$

When $P>Q$, using Corollary 2.7, it follows that

$$\sum_{|\gamma_\pi|\leqslant T}\bigg(\frac{Q}{P}\bigg)^{i\gamma_\pi} = -\frac{T}{\pi}\sqrt{\frac{Q}{P}}\Lambda_\pi\bigg(\frac{P}{Q}\bigg)+O_{\pi,d,\xi,\varepsilon}((PQ)^{\frac12+\theta_m+\varepsilon}\log^2 T).$$

Note that $\Lambda_\pi(P/Q)\neq 0$ if and only if P/Q is a prime power. Hence we deduce that

$$\mathcal{A}_{k,\ell}(P>Q) \ll_{\pi,d,\alpha,\xi,\varepsilon} \frac{1}{\log T}\bigg(\sum_{p\leqslant V}\frac{(a_p\Lambda_\pi(p))^{k-\ell}\Lambda_\pi(p^{k-\ell})}{p^{k-\ell}}\bigg)\bigg(\sum_{q\leqslant V}\frac{a_q^2|\Lambda_\pi(q)|^2}{q}\bigg)^\ell + T^{\eta-1+\varepsilon}.$$

Since $k-\ell\geqslant 2$, the sum over p is $O_{\pi,d,\alpha,\xi}(1)$ following the same argument how we established (3.4). The sum over q is $\ll_{\pi,d,\alpha,\xi} (\log X)^\ell$ by Lemma 2.10. Therefore choosing $\varepsilon>0$ sufficiently small, we obtain

$$\mathcal{A}_{k,\ell}(P>Q) \ll_{\pi,d,\alpha,\xi} (\log T)^{\ell-1}. \tag{3.6}$$

In conclusion, from (3.4), (3.5) and (3.6), the overall contribution in Case 1 is $\ll_{\pi,d,\alpha,\xi} (\log T)^{d-2}$. A concomitant argument follows when $\ell>k$.

Case 2 : When $k=\ell$. For $k=\ell=d$, we have

$$\mathcal{A}_{d,d} = \frac{1}{N_\pi(T)}\sum_{\substack{p_1,p_2,\ldots,p_d\leqslant V\\ q_1,q_2,\ldots,q_d\leqslant V}}\frac{(\prod_{i=1}^d a_{p_i}\Lambda_\pi(p_i))(\prod_{j=1}^d a_{q_j}\overline{\Lambda_\pi(q_j)})}{(PQ)^{\frac12}}\sum_{|\gamma_\pi|\leqslant T}\bigg(\frac{Q}{P}\bigg)^{i\gamma_\pi}, \tag{3.7}$$

where $P=p_1p_2\cdots p_d$ and $Q=q_1q_2\cdots q_d$. The main terms occur when $P=Q$. If all the p_i's are distinct, there are exactly $d!$ ways to rewrite P as Q. Applying Lemma 2.10 and Lemma 2.11, the contribution from the case $P = Q$ is

$$\sum_{\substack{p_1,p_2,\ldots,p_d\leqslant V\\ q_1,q_2,\ldots,q_d\leqslant V\\ P=Q}} \frac{(\prod_{i=1}^d a_{p_i}\Lambda_\pi(p_i))(\prod_{j=1}^d a_{q_j}\overline{\Lambda_\pi(q_j)})}{P}$$

$$= d! \sum_{p_1,p_2,\ldots,p_d\leqslant V} \frac{(\prod_{i=1}^d a_{p_i}^2|\Lambda_\pi(p_i)|^2)}{P}$$

$$- \sum_{j=2}^{d}\left\{ d!\Big(1-\frac{1}{j!}\Big)\Big(\sum_{p\leqslant V}\frac{a_p^{2j}|\Lambda_\pi(p)|^{2j}}{p^j}\Big) \sum_{\substack{p_1,p_2,\ldots,p_{d-j}\leqslant V\\ p_i\neq p}} \frac{\prod_{i=1}^{d-j} a_{p_i}^2|\Lambda_\pi(p_i)|^2}{p_1p_2\cdots p_{d-j}}\right\}$$

$$= d!\Big(\sum_{p\leqslant V}\frac{a_p^2|\Lambda_\pi(p)|^2}{p}\Big)^d + O_{\pi,d,\alpha,\xi}((\log T)^{d-2}). \tag{3.8}$$

Observe that the error term in (3.8) doesn't exist when $d = 1$.

Now, we concentrate on the error terms, which occur when $P \neq Q$. Here, Q/P or P/Q is never a prime power. Applying Corollaries 2.6 and 2.7, we see that

$$\mathcal{A}_{d,d}(P > Q) + \mathcal{A}_{d,d}(P < Q) \ll_{\pi,d,\alpha,\xi,\varepsilon} T^{\eta-1+\varepsilon}. \tag{3.9}$$

Therefore choosing $\varepsilon > 0$ sufficiently small, from (3.8) and (3.9), the overall contribution in Case 2 is

$$d!\Big(\sum_{p\leqslant V}\frac{a_p^2|\Lambda_\pi(p)|^2}{p}\Big)^d + O_{\pi,d,\alpha,\xi}(\max\{1,(\log T)^{d-2}\}).$$

Combining the two cases, we arrive at our desired conclusion. □

Our next lemma follows directly from Lemma 3.1 combined with Lemma 2.8.

Lemma 3.2. *Let $m \in \mathbb{N}, \pi \in \mathscr{A}_m$ and $\theta_m \in [0, \frac{1}{2} - \frac{1}{m^2+1}]$ be an admissible exponent towards the Ramanujan conjecture for $L(s,\pi)$. Assume Hypothesis $\mathbf{H}_\pi$ and the Riemann Hypothesis for $L(s,\pi)$. Let $d \in \mathbb{N}$, $T \geqslant 2$ and α, ξ, X, V be as in* (3.1) *satisfying*

$$\alpha < \frac{1}{2dm(1+\xi\theta_m)}.$$

Let $\{a_n\}$ be the sequence defined by (2.26). *Then*

$$\frac{1}{N_\pi(T)}\sum_{|\gamma_\pi|\leqslant T}\Big(\mathrm{Re}\sum_{p\leqslant V}\frac{a_p\Lambda_\pi(p)}{p^{\frac{1}{2}+i\gamma_\pi}}\Big)^{2d} = \boldsymbol{\mu}_{2d}\Big(\frac{1}{2}\sum_{p\leqslant V}\frac{a_p^2|\Lambda_\pi(p)|^2}{p}\Big)^d + O_{\pi,d,\alpha,\xi}(\max\{1,(\log T)^{d-2}\}).$$

where $\boldsymbol{\mu}_{2d}$ is defined by (1.19). *Also, if*

$$\alpha < \frac{1}{m(2d-1)(1+\xi\theta_m)},$$

then we have

$$\frac{1}{N_\pi(T)}\sum_{|\gamma_\pi|\leqslant T}\Big(\mathrm{Re}\sum_{p\leqslant V}\frac{a_p\Lambda_\pi(p)}{p^{\frac{1}{2}+i\gamma_\pi}}\Big)^{2d-1} \ll_{\pi,d,\alpha,\xi} (\log T)^{d-1}.$$

Proof. All implied constants in the proof may depend at most on π, d, α and ξ. First consider the case when the exponent is even. Using Lemma 2.8, we have

$$\frac{1}{N_\pi(T)}\sum_{|\gamma_\pi|\leqslant T}\Big(\mathrm{Re}\sum_{p\leqslant V}\frac{a_p\Lambda_\pi(p)}{p^{\frac{1}{2}+i\gamma_\pi}}\Big)^{2d} = \frac{1}{2^{2d}}\binom{2d}{d}\mathrm{Re}(\mathcal{A}_{d,d}) + \frac{1}{2^{2d-1}}\sum_{i=0}^{d-1}\binom{2d}{i}\mathrm{Re}(\mathcal{A}_{2d-i,i}),$$

where $\mathcal{A}_{k,\ell}$ is given by (3.2). An application of Lemma 3.1 shows that

$$\frac{1}{N_\pi(T)}\sum_{|\gamma_\pi|\leqslant T}\bigg(\operatorname{Re}\sum_{p\leqslant V}\frac{a_p\Lambda_\pi(p)}{p^{\frac12+i\gamma_\pi}}\bigg)^{2d}=\frac{d!}{2^{2d}}\binom{2d}{d}\bigg(\sum_{p\leqslant V}\frac{a_p^2|\Lambda_\pi(p)|^2}{p}\bigg)^d+O_{\pi,d,\alpha,\xi}(\max\{1,(\log T)^{d-2}\})$$
$$=\boldsymbol{\mu}_{2d}\bigg(\frac12\sum_{p\leqslant V}\frac{a_p^2|\Lambda_\pi(p)|^2}{p}\bigg)^d+O_{\pi,d,\alpha,\xi}(\max\{1,(\log T)^{d-2}\}).$$

When the exponent is odd, in Lemma 2.8 none of the terms $z^i\overline{z}^j$ are such that $i=j$. Following the proof of Lemma 3.1 and writing $k+\ell=2d-1$, the contributions arise only from Case 1. Assuming $k>\ell$, the critical situation is when $k-\ell=1$. The bounds corresponding to (3.4) and (3.6) are then amplified by an extra factor of $\log T$ following an application of Lemma 2.10. A similar argument holds when $\ell>k$. □

Now we extend our sums to prime powers. Let $m\in\mathbb{N}$ and $\pi\in\mathscr{A}_m$. Let $X,T\geqslant 2$ and $V=X^\xi$ for some fixed $\xi>1$. Let $\{a_n\}$ be the sequence as in (2.26). Define

$$\operatorname{Av}_\pi(T,X,V,\{a_n\}):=\frac{1}{N_\pi(T)}\sum_{|\gamma_\pi|\leqslant T}\operatorname{Re}\sum_{n\leqslant V}\frac{a_n\Lambda_\pi(n)}{n^{\rho_\pi}} \tag{3.10}$$

$$\text{and}\quad \operatorname{Var}_\pi(T,X,V,\{a_n\}):=\frac12\sum_{n\leqslant V}\frac{a_n^2|\Lambda_\pi(n)|^2}{n}-\operatorname{Av}_\pi^2(T,X,V,\{a_n\}). \tag{3.11}$$

We first prove asymptotic formulas for Av_π and Var_π without assuming RH for $L(s,\pi)$.

Lemma 3.3. *Let $m\in\mathbb{N},\pi\in\mathscr{A}_m$ and $\theta_m\in[0,\frac12-\frac{1}{m^2+1}]$ be an admissible exponent towards the Ramanujan conjecture for $L(s,\pi)$. Assume Hypothesis $\mathbf{H}_\pi$ for $L(s,\pi)$. Suppose $T\geqslant 2$ and α,ξ,X,V be as in* (3.1) *satisfying*

$$\alpha<\frac{1}{m(1+\xi\theta_m)}.$$

Let $\{a_n\}$ be the sequence defined by (2.26). *Then*

$$\operatorname{Av}_\pi(T,X,V,\{a_n\})=-\alpha+O_{\pi,\alpha,\xi}\left(\frac{1}{\log T}\right), \tag{3.12}$$

$$\textit{and}\quad \operatorname{Var}_\pi(T,X,V,\{a_n\})=\frac{\alpha m\log T}{4}+O_{\pi,\alpha,\xi}(1). \tag{3.13}$$

Proof. All implied constants in the proof may depend at most on π,α and ξ. By Corollary 2.7, we have

$$\operatorname{Av}_\pi=-\frac{T}{\pi N_\pi(T)}\sum_{n\leqslant V}\frac{a_n|\Lambda_\pi(n)|^2}{n}+O_{\pi,\alpha,\xi,\varepsilon}(T^{\alpha m(1+\eta\theta_m)-1+\varepsilon}), \tag{3.14}$$

for any $\varepsilon>0$. By Lemma 2.10,

$$\sum_{n\leqslant V}\frac{a_n|\Lambda_\pi(n)|^2}{n}=\alpha m\log T+O_{\pi,\alpha,\xi}(1). \tag{3.15}$$

We choose $\varepsilon>0$ sufficiently small depending on π,α and ξ. Substituting (3.15) into (3.14) and applying a more precise form of (2.8) (see [22, Theorem 5.8]), (3.12) follows. Now, using (3.12) and Lemma 2.10 in (3.11), (3.13) holds. □

We are now ready to estimate our moments involving sums over prime powers.

Lemma 3.4. *Let $m \in \mathbb{N}, \pi \in \mathscr{A}_m$ and $\theta_m \in [0, \frac{1}{2} - \frac{1}{m^2+1}]$ be an admissible exponent towards the Ramanujan conjecture for $L(s, \pi)$. Assume Hypothesis $\mathbf{H}_\pi$ and the Riemann Hypothesis for $L(s, \pi)$. Let $d \in \mathbb{N}$, $T \geqslant 2$ and α, ξ, X, V be as in* (3.1) *satisfying*

$$\alpha < \frac{1}{2dm(1 + \xi\theta_m)}.$$

Let $\{a_n\}$ be the sequence defined by (2.26). *Then*

$$\frac{1}{N_\pi(T)} \sum_{|\gamma_\pi| \leqslant T} \left(\operatorname{Re} \sum_{n \leqslant V} \frac{a_n \Lambda_\pi(n)}{n^{\frac{1}{2}+i\gamma_\pi}} \right)^{2d} = \boldsymbol{\mu}_{2d} \operatorname{Var}_\pi^d + O_{\pi,d,\alpha,\xi}((\log T)^{d-\frac{1}{2}}),$$

where $\boldsymbol{\mu}_{2d}$ is given by (1.19) *and* Var_π *is defined by* (3.11). *Also, if*

$$\alpha < \frac{1}{m(2d-1)(1 + \xi\theta_m)},$$

then we have

$$\frac{1}{N_\pi(T)} \sum_{|\gamma_\pi| \leqslant T} \left(\operatorname{Re} \sum_{n \leqslant V} \frac{a_n \Lambda_\pi(n)}{n^{\frac{1}{2}+i\gamma_\pi}} \right)^{2d-1} \ll_{\pi,d,\alpha,\xi} (\log T)^{d-1}.$$

Proof. All implied constants in the proof may depend at most on π, d, α and ξ. First consider the case when the exponent is even. We write

$$\frac{1}{N_\pi(T)} \sum_{|\gamma_\pi| \leqslant T} \left(\operatorname{Re} \sum_{n \leqslant V} \frac{a_n \Lambda_\pi(n)}{n^{\frac{1}{2}+i\gamma_\pi}} \right)^{2d}$$

$$= \frac{1}{N_\pi(T)} \sum_{j=0}^{2d} \binom{2d}{j} \sum_{|\gamma_\pi| \leqslant T} \left(\operatorname{Re} \sum_{p \leqslant V} \frac{a_p \Lambda_\pi(p)}{p^{\frac{1}{2}+i\gamma_\pi}} \right)^j \left(\operatorname{Re} \sum_{\substack{n \leqslant V \\ n=p^k, k \geqslant 2}} \frac{a_n \Lambda_\pi(n)}{n^{\frac{1}{2}+i\gamma_\pi}} \right)^{2d-j}. \tag{3.16}$$

By Lemma 3.2, the contribution to (3.16) from $j = 2d$ is

$$\boldsymbol{\mu}_{2d} \left(\frac{1}{2} \sum_{p \leqslant V} \frac{a_p^2 |\Lambda_\pi(p)|^2}{p} \right)^d + O_{\pi,d,\alpha,\xi}(\max\{1, (\log T)^{d-2}\}) \tag{3.17}$$

where $\boldsymbol{\mu}_{2d}$ is defined by (1.19). By Hypothesis $\mathbf{H}_\pi$ (see [47, Eq. 2.25]),

$$\operatorname{Var}_\pi = \frac{1}{2} \sum_{p \leqslant V} \frac{a_p^2 |\Lambda_\pi(p)|^2}{p} + O_{\pi,\eta}(1).$$

Therefore applying Lemma 3.3, (3.17) is equal to

$$\boldsymbol{\mu}_{2d} \operatorname{Var}_\pi^d + O_{\pi,d,\alpha,\xi}((\log T)^{d-1}).$$

For $j = 0$, note that

$$\left| \sum_{\substack{n \leqslant V \\ n=p^k, k \geqslant 2}} \frac{a_n \Lambda_\pi(n)}{n^{\frac{1}{2}+i\gamma_\pi}} \right|^{2d} \ll_d \left| \sum_{\substack{n=p^k \leqslant V \\ 2 \leqslant k \leqslant m^2+1}} \frac{a_{p^k} \Lambda_\pi(p^k)}{p^{\frac{k}{2}+ki\gamma_\pi}} \right|^{2d} + \left| \sum_{\substack{n \leqslant V \\ n=p^k, k > m^2+1}} \frac{a_n \Lambda_\pi(n)}{n^{\frac{1}{2}+i\gamma_\pi}} \right|^{2d}.$$

Since

$$\sum_{k > m^2+1} \sum_{\substack{n \leqslant V \\ n=p^k}} \frac{a_n \Lambda_\pi(n)}{n^{\frac{1}{2}+i\gamma_\pi}} \ll_{\pi,\xi} 1,$$

we obtain

$$\frac{1}{N_\pi(T)}\sum_{|\gamma_\pi|\leqslant T}\Bigg(\mathrm{Re}\sum_{\substack{n\leqslant V\\ n=p^k,k\geqslant 2}}\frac{a_n\Lambda_\pi(n)}{n^{\frac12+i\gamma_\pi}}\Bigg)^{2d} \ll_{\pi,d,\alpha,\xi} \frac{1}{N_\pi(T)}\sum_{|\gamma_\pi|\leqslant T}\Bigg|\sum_{\substack{n=p^k\leqslant V\\ 2\leqslant k\leqslant m^2+1}}\frac{a_{p^k}\Lambda_\pi(p^k)}{p^{\frac{k}{2}+ki\gamma_\pi}}\Bigg|^{2d}+1$$

$$\ll_{\pi,d,\alpha,\xi} \sum_{k=2}^{m^2+1}\frac{1}{N_\pi(T)}\sum_{|\gamma_\pi|\leqslant T}\Bigg|\sum_{n=p^k\leqslant V}\frac{a_{p^k}\Lambda_\pi(p^k)}{p^{\frac{k}{2}+ki\gamma_\pi}}\Bigg|^{2d}+1, \tag{3.18}$$

where in the last step, we apply Jensen's inequality. To bound the contribution from the prime powers with $2\leqslant k\leqslant m^2+1$, we follow the same approach as in Case 2 of Lemma 3.1. Fixing some k and opening the $2d$-th power, we obtain

$$\frac{1}{N_\pi(T)}\sum_{\substack{p_1,p_2,\dots,p_d\leqslant V^{1/k}\\ q_1,q_2,\dots,q_d\leqslant V^{1/k}}}\frac{(\prod_{i=1}^d a_{p_i^k}\Lambda_\pi(p_i^k))(\prod_{j=1}^d a_{q_j^k}\overline{\Lambda_\pi(q_j^k)})}{(PQ)^{k/2}}\sum_{|\gamma_\pi|\leqslant T}\left(\frac{Q}{P}\right)^{ki\gamma_\pi},$$

where $P=p_1p_2\cdots p_d$ and $Q=q_1q_2\cdots q_d$. The contribution from the diagonal terms $P=Q$ is $O_{\pi,d,\alpha,\xi}(1)$ by Hypothesis $\mathbf{H}_\pi$. For the non-diagonal terms $P\neq Q$, note that Q/P is never a prime power and thus by Corollaries 2.6 and 2.7, their contribution is also $O_{\pi,d,\alpha,\xi}(1)$. Therefore the overall contribution when $j=0$ is $O_{\pi,d,\alpha,\xi}(1)$. Now suppose $j\neq 0,2d$. By Hölder's inequality, the sum corresponding to each such j is

$$\ll_{\pi,d,\alpha,\xi}\frac{1}{N_\pi(T)}\Bigg(\sum_{|\gamma_\pi|\leqslant T}\Bigg(\mathrm{Re}\sum_{p\leqslant V}\frac{a_p\Lambda_\pi(p)}{p^{\frac12+i\gamma_\pi}}\Bigg)^{2d}\Bigg)^{\frac{j}{2d}}\Bigg(\sum_{|\gamma_\pi|\leqslant T}\Bigg(\mathrm{Re}\sum_{\substack{n\leqslant V\\ n=p^k,k\geqslant 2}}\frac{a_n\Lambda_\pi(n)}{n^{\frac12+i\gamma_\pi}}\Bigg)^{2d}\Bigg)^{\frac{2d-j}{2d}}.$$

Using the estimates for $j=0,2d$, the above is $\ll_{\pi,d,\alpha,\xi}(\log T)^{d-\frac12}$. Combining the estimates for different values of j in (3.16), we arrive at our desired result when the exponent is even.

Now, suppose the exponent is odd. We consider two cases. When $d=1$, by definition and Lemma 3.3,

$$\frac{1}{N_\pi(T)}\sum_{|\gamma_\pi|\leqslant T}\mathrm{Re}\sum_{n\leqslant V}\frac{a_n\Lambda_\pi(n)}{n^{\frac12+\gamma_\pi}}=\mathrm{Av}_\pi\ll_{\pi,\alpha,\xi}1.$$

Now assume $d>1$. The proof is similar to the case for even exponents. For $j=2d-1$, by Lemma 3.2,

$$\frac{1}{N_\pi(T)}\sum_{|\gamma_\pi|\leqslant T}\Bigg(\mathrm{Re}\sum_{p\leqslant V}\frac{a_p\Lambda_\pi(p)}{p^{\frac12+i\gamma_\pi}}\Bigg)^{2d-1}\ll_{\pi,d,\alpha,\xi}(\log T)^{d-1}.$$

For $j=0$, by Cauchy–Schwarz and our previous case on prime powers for even exponents,

$$\frac{1}{N_\pi(T)}\sum_{|\gamma_\pi|\leqslant T}\Bigg(\mathrm{Re}\sum_{\substack{n\leqslant V\\ n=p^k,k\geqslant 2}}\frac{a_n\Lambda_\pi(n)}{n^{\frac12+i\gamma_\pi}}\Bigg)^{2d-1}\ll_{\pi,d,\alpha,\xi}1.$$

When $j\neq 0$ or $2d-1$, we first use Hölder's inequality to bound the contribution by

$$\frac{1}{N_\pi(T)}\Bigg(\sum_{|\gamma_\pi|\leqslant T}\Bigg|\mathrm{Re}\sum_{p\leqslant V}\frac{a_p\Lambda_\pi(p)}{p^{\frac12+i\gamma_\pi}}\Bigg|^{2d-1}\Bigg)^{\frac{j}{2d-1}}\Bigg(\sum_{|\gamma_\pi|\leqslant T}\Bigg|\mathrm{Re}\sum_{\substack{n\leqslant V\\ n=p^k,k\geqslant 2}}\frac{a_n\Lambda_\pi(n)}{n^{\frac12+i\gamma_\pi}}\Bigg|^{2d-1}\Bigg)^{\frac{2d-1-j}{2d-1}}. \tag{3.19}$$

For the sum over primes, by Cauchy–Schwarz and Lemma 3.2, one has

$$\begin{aligned}
&\sum_{|\gamma_\pi|\leqslant T}\left|\operatorname{Re}\sum_{p\leqslant V}\frac{a_p\Lambda_\pi(p)}{p^{\frac{1}{2}+i\gamma_\pi}}\right|^{2d-1} \\
&\qquad \ll_{\pi,d,\alpha,\xi}\left(\sum_{|\gamma_\pi|\leqslant T}\left(\operatorname{Re}\sum_{p\leqslant V}\frac{a_p\Lambda_\pi(p)}{p^{\frac{1}{2}+i\gamma_\pi}}\right)^{2d}\right)^{\frac{1}{2}}\left(\sum_{|\gamma_\pi|\leqslant T}\left(\operatorname{Re}\sum_{p\leqslant V}\frac{a_p\Lambda_\pi(p)}{p^{\frac{1}{2}+i\gamma_\pi}}\right)^{2d-2}\right)^{\frac{1}{2}} \\
&\qquad \ll_{\pi,d,\alpha,\xi} (\log T)^{d-\frac{1}{2}}, \qquad (3.20)
\end{aligned}$$

A similar argument works for the sum over higher prime powers and thus, (3.19) is

$$\ll_{\pi,d,\alpha,\xi}\frac{1}{N_\pi(T)}(N_\pi(T)(\log X)^{d-\frac{1}{2}})^{\frac{j}{2d-1}}(N_\pi(T))^{\frac{2d-1-j}{2d-1}}\ll_{\pi,d,\alpha,\xi}(\log T)^{d-1},$$

when $j\neq 0, 2d-1$. This completes the proof. □

4. Proofs of Theorems 1.4 and 1.5

In this section, our goal is to prove Theorems 1.4 and 1.5. Let's slightly adjust (3.1). We let $\xi>\frac{4}{3}$. This additional constraint is required for our forthcoming arguments in this section. See Remark 4.2 for more on this. We begin with the following lemma.

Lemma 4.1. *Let $m\in\mathbb{N},\pi\in\mathscr{A}_m$ and $\theta_m\in[0,\frac{1}{2}-\frac{1}{m^2+1}]$ be an admissible exponent towards the Ramanujan conjecture for $L(s,\pi)$. Assume Hypothesis $\mathbf{H}_\pi$ and the Riemann Hypothesis for $L(s,\pi)$. Let $T\geqslant 2$ and α,ξ,X,V be as in* (3.1) *satisfying*

$$\alpha<\frac{1}{m(1+\xi\theta_m)}\quad\text{and}\quad \xi>\frac{4}{3}.$$

Let $\{a_n\}$ be the sequence defined by (2.26). *Then*

$$\operatorname{Re}\mathcal{S}_{\mathrm{av},\pi}(X,T)=\alpha+O_{\pi,\alpha,\xi}\left(\frac{1}{\log T}\right).$$

Proof. Applying Lemma 2.4, we can write

$$\begin{aligned}
\operatorname{Re}\mathcal{S}_{\mathrm{av},\pi}(X,T)&=\frac{1}{N_\pi(T)}\sum_{|\gamma_{\pi'}|\leqslant T}\operatorname{Re}\mathcal{S}_\pi(X,T,\rho'_\pi) \\
&=-\frac{1}{N_\pi(T)}\sum_{|\gamma'_\pi|\leqslant T}\operatorname{Re}\sum_{n=1}^{\infty}\frac{a_n\Lambda_\pi(n)}{n^{\frac{1}{2}+i\gamma'_\pi}}+\frac{1}{N_\pi(T)}\sum_{\rho'_\pi\in\mathcal{E}}\left(\operatorname{Re}\mathcal{S}_\pi(X,T,\rho'_\pi)+\operatorname{Re}\sum_{n=1}^{\infty}\frac{a_n\Lambda_\pi(n)}{n^{\frac{1}{2}+i\gamma'_\pi}}\right) \\
&\quad+O_\pi\left(\frac{\log T}{X^2}+\frac{\sqrt{X}}{T}+\frac{\log T}{\sqrt{T}}\right), \qquad (4.1)
\end{aligned}$$

where $\mathcal{E}$ is an exceptional set of non-trivial zeros $\rho'_\pi=\beta_\pi+i\gamma'_\pi$ of $L(s,\pi)$ with $|\gamma'_\pi|\leqslant T$ and $\#\mathcal{E}\ll_\pi\sqrt{T}\log T$. The second term on the right hand side of (4.1) is

$$\ll_{\pi,\alpha,\xi}\frac{1}{N_\pi(T)}\sum_{\rho'_\pi\in\mathcal{E}}(\log T+(X\log X)^{1/2})\ll_{\pi,\alpha,\xi}T^{-\delta_1}, \qquad (4.2)$$

for some $\delta_1=\delta_1(\pi,\alpha,\xi)>0$. To see this, we first remark that here we don't use any cancellation on the sum over zeros. By Cauchy–Schwarz and Lemma 2.10, the sum over n is $\ll_\pi (X\log X)^{1/2}$. To bound the expression $\operatorname{Re}\mathcal{S}(X,T,\rho'_\pi)$, we assume RH for $L(s,\pi)$ and apply standard estimates on sums over zeros (see [22, Eq. 5.32]). For the first term on the right hand side of (4.1), we write

$$\frac{1}{N_\pi(T)}\sum_{|\gamma'_\pi|\leqslant T}\operatorname{Re}\sum_{n=1}^{\infty}\frac{a_n\Lambda_\pi(n)}{n^{\frac12+i\gamma'_\pi}}-\mathrm{Av}_\pi=\frac{1}{N_\pi(T)}\sum_{|\gamma'_\pi|\leqslant T}\operatorname{Re}\sum_{n>V}\frac{a_n\Lambda_\pi(n)}{n^{\frac12+i\gamma'_\pi}}. \tag{4.3}$$

where $\mathrm{Av}_\pi=\mathrm{Av}_\pi(T,X,V,\{a_n\})$ is as defined in (3.10). Consider the inner sum over $n>V$ in the right hand side of (4.3). By an application of Cauchy–Schwarz and Lemma 2.9,

$$\sum_{n>V}\frac{a_n\Lambda_\pi(n)}{n^{\frac12}}\ll_{\pi,\alpha,\xi}\sum_{k=0}^{\infty}\Bigg(\sum_{n\in(2^kV,2^{k+1}V]}a_n^2\Bigg)^{\frac12}\Bigg(\sum_{n\in(2^kV,2^{k+1}V]}\frac{|\Lambda_\pi(n)|^2}{n}\Bigg)^{\frac12}\ll_{\pi,\alpha,\xi}T^{-\delta_2} \tag{4.4}$$

for some $\delta_2=\delta_2(\pi,\alpha,\xi)>0$. Therefore bounding the outer sum over γ'_π in the right hand side of (4.3) trivially, we arrive at

$$\frac{1}{N_\pi(T)}\sum_{|\gamma'_\pi|\leqslant T}\operatorname{Re}\sum_{n=1}^{\infty}\frac{a_n\Lambda_\pi(n)}{n^{\frac12+i\gamma'_\pi}}-\mathrm{Av}_\pi\ll_{\pi,\alpha,\xi}T^{-\delta_2}. \tag{4.5}$$

Putting together (4.2) and (4.5) into (4.1) and applying Lemma 3.3, the proof follows. □

Proof of Theorem 1.5. First suppose that the exponent is even and write $r=2d$. We let $X=T^{\alpha m}$ and choose $V=X^\xi$ for some $\xi=\xi(\pi,d,\alpha)>\frac43$ such that

$$\alpha<\frac{1}{2dm(1+\xi\theta_m)}.$$

Then Lemma 2.4, alongside the argument we used to establish (4.4), shows that for all non-trivial zeros $\rho'_\pi=\frac12+i\gamma'_\pi$ of $L(s,\pi)$ with $|\gamma'_\pi|\leqslant T$ outside an exceptional set of size at most $O_\pi(\sqrt{T}\log T)$, we have

$$\operatorname{Re}\mathcal{S}_\pi(X,T,\rho'_\pi)=-\operatorname{Re}\sum_{n\leqslant V}\frac{a_n\Lambda_\pi(n)}{n^{\frac12+i\gamma'_\pi}}+E'_\pi(X,T,\rho'_\pi), \tag{4.6}$$

where $E'_\pi(X,T,\rho'_\pi)\ll_{\pi,d,\alpha}T^{-\delta_1}$ for some $\delta_1=\delta_1(\pi,d,\alpha)>0$. Therefore we have

$$\begin{aligned}&\frac{1}{N_\pi(T)}\sum_{|\gamma'_\pi|\leqslant T}(\operatorname{Re}\mathcal{S}_\pi(X,T,\rho'_\pi))^{2d}\\&\quad=\frac{1}{N_\pi(T)}\sum_{|\gamma'_\pi|\leqslant T}\Bigg(-\operatorname{Re}\sum_{n\leqslant V}\frac{a_n\Lambda_\pi(n)}{n^{\frac12+i\gamma'_\pi}}+E'_\pi(X,T,\rho'_\pi)\Bigg)^{2d}\\&\quad\quad+\frac{1}{N_\pi(T)}\sum_{\rho'_\pi\in\mathcal{E}}\Bigg((\operatorname{Re}\mathcal{S}_\pi(X,T,\rho'_\pi))^{2d}-\Bigg(\operatorname{Re}\sum_{n\leqslant V}\frac{a_n\Lambda_\pi(n)}{n^{\frac12+i\gamma'_\pi}}+E'_\pi(X,T,\rho'_\pi)\Bigg)^{2d}\Bigg)\\&\quad=\mathcal{U}_1+\mathcal{U}_2\end{aligned}$$

say, where $\mathcal{E}$ is an exceptional set of non-trivial zeros $\rho'_\pi=\beta_\pi+i\gamma'_\pi$ of $L(s,\pi)$ with $|\gamma'_\pi|\leqslant T$ and $\#\mathcal{E}\ll_\pi\sqrt{T}\log T$. An application of Lemma 3.4 shows that

$$\begin{aligned}\mathcal{U}_1&=\frac{1}{N_\pi(T)}\sum_{|\gamma'_\pi|\leqslant T}\Bigg(\operatorname{Re}\sum_{n\leqslant V}\frac{a_n\Lambda_\pi(n)}{n^{\frac12+i\gamma'_\pi}}\Bigg)^{2d}+O_{\pi,d,\alpha}\Bigg(\frac{(\log T)^{d-1}}{T^{\delta_1}}\Bigg)\\&=\boldsymbol{\mu}_{2d}\mathrm{Var}_\pi^d+O_{\pi,d,\alpha}((\log T)^{d-\frac12})\end{aligned}$$

where $\boldsymbol{\mu}_{2d}$ is given by (1.19) and Var_π is as defined in (3.11). An argument similar to how we proved (4.2) shows that $\mathcal{U}_2 \ll_{\pi,d,\alpha} T^{-\delta_2}$ for some $\delta_2 = \delta_2(\pi, d, \alpha) > 0$. Combining the estimates for $\mathcal{U}_1$ and $\mathcal{U}_2$,

$$\frac{1}{N_\pi(T)} \sum_{|\gamma'_\pi| \leqslant T} (\mathrm{Re}\, \mathcal{S}_\pi(X, T, \rho'_\pi))^{2d} = \boldsymbol{\mu}_{2d} \mathrm{Var}_\pi^d + O_{\pi,d,\alpha}((\log T)^{d-\frac{1}{2}}). \tag{4.7}$$

Applying Lemmata 3.3 and 4.1, it follows that

$$\mathcal{M}_{\pi,2d}(T^{\alpha m}, T) = \boldsymbol{\mu}_{2d} \left(\frac{\alpha m \log T}{4} \right)^d + O_{\pi,\alpha,d}((\log T)^{d-\frac{1}{2}}).$$

When the exponent is odd, the proof is similar except that $\mathcal{U}_1 \ll_{\pi,d,\alpha} (\log T)^{d-1}$. □

Remark 4.2. The additional restriction $\xi > \frac{4}{3}$, compared to the results in Section 3, is crucial for establishing (4.6). While one may improve the results for (4.4) by leveraging the outer sum over zeros, (4.6) critically depends on the choice $\xi > \frac{4}{3}$. This dependence arises because the sequence $\{a_n\}$ defined by (2.26), originating from the weight function $w(u)$ in (1.4), does not decay rapidly enough. This issue does not occur when using smooth, compactly supported weight functions in Section 7.

Proof of Theorem 1.4. For each fixed $r \in \mathbb{N}$, there exists $J_r \in \mathbb{N}$ such that

$$\alpha_j < \frac{1}{rm(1 + \frac{4}{3}\theta_m)}$$

for all $j \geqslant J_r$. By Theorem 1.5, for each such α_j, there exists $U_{j,r}(\pi, \alpha_j) > 0$ depending on π and α_j such that for any $T_j \geqslant U_{j,r}$, we have

$$\frac{1}{N_\pi(T_j)} \sum_{|\gamma'_\pi| \leqslant T_j} \left(\frac{\alpha_j m \log T_j}{4} \right)^{-\frac{r}{2}} \left(\mathrm{Re}\, \mathcal{S}_\pi(T_j^{\alpha_j m}, T_j, \rho'_\pi) - \alpha_j \right)^r = \boldsymbol{\mu}_r + O_{\pi,r}((\log T_j)^{-\frac{1}{2}}),$$

where $\boldsymbol{\mu}_r$ is defined by (1.19). Define the sequence $\{V_j\}_{j \in \mathbb{N}}$ as follows. For all $j < J_1$, let $V_j = 3$. As j varies over $\mathbb{N}$, we choose

$$V_j = \max\{U_{j,1}, U_{j,2}, \cdots, U_{j,r}\}, \quad J_r \leqslant j < J_{r+1}.$$

Then for any sequence $\{T_j\}_{j \in \mathbb{N}}$ such that $T_j \geqslant V_j$ for all $j \in \mathbb{N}$, and for every positive integer r, we have

$$\frac{1}{N_\pi(T_j)} \sum_{|\gamma'_\pi| \leqslant T_j} \left(\frac{\alpha_j m \log T_j}{4} \right)^{-\frac{r}{2}} \left(\mathrm{Re}\, \mathcal{S}_\pi(T_j^{\alpha_j m}, T_j, \rho'_\pi) - \alpha_j \right)^r \to \boldsymbol{\mu}_r, \ \text{ as } \ j \to \infty. \tag{4.8}$$

The conclusion of Theorem 1.4 now follows by employing the moment method along with an application of the Berry–Esseen theorem. This is a well-known argument, for example, see Davenport–Erdős [6], Montgomery–Soundararajan [42], Basak–Nath–Zaharescu [2] for such applications. □

5. Moments of Sums over Primes: Part II

Let $m \in \mathbb{N}, \pi \in \mathscr{A}_m$ and $\widetilde{\mathcal{S}}_\pi(X, T, \rho'_\pi)$ be as defined in (1.7). In this section, we focus on obtaining results about the distribution of $\widetilde{\mathcal{S}}_\pi(X, T, \rho'_\pi)$ and thereby addressing the issue of uniformity of α with respect to T. To do so, we assume the following zero density hypothesis for $L(s, \pi)$ which we alluded to in Section 1.

Hypothesis $\mathbf{Z}_\pi$: There exist constants $A_\pi > 0$ (depending on π) such that

$$N_\pi(\sigma, T) = \#\{\rho_\pi = \beta_\pi + i\gamma_\pi : \sigma \leqslant \beta_\pi, |\gamma_\pi| \leqslant T, L(s, \rho_\pi) = 0\} \ll_\pi T^{1 - A_\pi(\sigma - \frac{1}{2})} \log T,$$

uniformly for $\sigma \geqslant \frac{1}{2}$ and $T \geqslant 2$.

The following proposition is crucial towards establishing results without assuming RH.

Proposition 5.1. *Let $m \in \mathbb{N}$, $\pi \in \mathscr{A}_m$ and assume Hypothesis $\mathbf{Z}_\pi$. Suppose $T, V \geqslant 2$ satisfying*

$$\log V = o\left(\frac{\log T}{\log\log T}\right). \tag{5.1}$$

Let $k, \ell \in \mathbb{N}$ and suppose $\{p_1, p_2, \dots, p_k\}$ and $\{q_1, q_2, \dots, q_\ell\}$ be two sets of primes satisfying $2 \leqslant p_i, q_j \leqslant V$ for all $1 \leqslant i \leqslant k$ and $1 \leqslant j \leqslant \ell$. Let $P = \prod_{i=1}^k p_i$ and $Q = \prod_{j=1}^\ell q_j$. Then

$$\sum_{|\gamma_\pi| \leqslant T} (PQ)^{\beta_\pi - \frac{1}{2}} \left(\frac{Q}{P}\right)^{i\gamma_\pi} = \sum_{|\gamma_\pi| \leqslant T} \left(\frac{Q}{P}\right)^{\rho_\pi - \frac{1}{2}} + O_{\pi,k,\ell}\left(\frac{T \log^2 V}{\log T}\right).$$

Proof. Let

$$H_1 = \sum_{|\gamma_\pi| \leqslant T} (PQ)^{\beta_\pi - \frac{1}{2}} \left(\frac{Q}{P}\right)^{i\gamma_\pi}, \quad H_2 = \sum_{|\gamma_\pi| \leqslant T} \left(\frac{Q}{P}\right)^{i\gamma_\pi} \quad \text{and} \quad H_3 = \sum_{|\gamma_\pi| \leqslant T} \left(\frac{Q}{P}\right)^{\rho_\pi - \frac{1}{2}}.$$

Choose $\delta = (K \log\log T)/\log T$ for some $K = K(A_\pi)$ sufficiently large depending only on A_π. We write

$$|H_1 - H_2| \leqslant \Bigg| \sum_{\substack{|\gamma_\pi| \leqslant T \\ |\beta_\pi - \frac{1}{2}| < \delta}} ((PQ)^{\beta_\pi - \frac{1}{2}} - 1)\left(\frac{Q}{P}\right)^{i\gamma_\pi} + \sum_{\substack{|\gamma_\pi| \leqslant T \\ |\beta_\pi - \frac{1}{2}| \geqslant \delta}} ((PQ)^{\beta_\pi - \frac{1}{2}} - 1)\left(\frac{P}{Q}\right)^{i\gamma_\pi} \Bigg|$$

$$\ll_{\pi,k,\ell} \sum_{\substack{|\gamma_\pi| \leqslant T \\ \frac{1}{2} < \beta_\pi < \frac{1}{2} + \delta}} ((PQ)^{\beta_\pi - \frac{1}{2}} + (PQ)^{\frac{1}{2} - \beta_\pi} - 2) + \sum_{\substack{|\gamma_\pi| \leqslant T \\ \beta_\pi \geqslant \frac{1}{2} + \delta}} (PQ)^{\delta} + N_\pi(\frac{1}{2} + \delta, T). \tag{5.2}$$

By Hypothesis $\mathbf{Z}_\pi$, our choice of δ and (5.1), we can allow

$$\sum_{\substack{|\gamma_\pi| \leqslant T \\ \beta_\pi \geqslant \frac{1}{2} + \delta}} (PQ)^{\delta} + N_\pi(\frac{1}{2} + \delta, T) \ll_{\pi,k,\ell} \frac{T}{(\log T)^{100}}, \tag{5.3}$$

say. So we focus on the first term on the right hand side of (5.2). By Taylor expansion, this is

$$\ll_{\pi,k,\ell} \log^2 V \sum_{\substack{|\gamma_\pi| \leqslant T \\ \frac{1}{2} < \beta_\pi < \frac{1}{2} + \delta}} (\beta_\pi - \frac{1}{2})^2. \tag{5.4}$$

By partial summation and Hypothesis $\mathbf{Z}_\pi$,

$$\sum_{\substack{|\gamma_\pi| \leqslant T \\ \frac{1}{2} < \beta_\pi < \frac{1}{2} + \delta}} (\beta_\pi - \frac{1}{2})^2 \ll_{\pi,k,\ell} T \log T \int_0^\delta \theta T^{-A_\pi \theta}\, \mathrm{d}\theta + \frac{T}{(\log T)^{100}} \ll_{\pi,k,\ell} \frac{T}{\log T}.$$

Therefore along with (5.2) and (5.3), we see that

$$|H_1 - H_2| \ll_{\pi,k,\ell} \frac{T \log^2 V}{\log T}. \tag{5.5}$$

Note that

$$\left(\frac{Q}{P}\right)^{i\gamma_\pi} = \left(\frac{Q}{P}\right)^{\rho_\pi - \frac{1}{2}} + \left(\frac{Q}{P}\right)^{i\gamma_\pi}\left(1 - \left(\frac{Q}{P}\right)^{\beta_\pi - \frac{1}{2}}\right),$$

and hence using techniques similar to how we established (5.5), we deduce that

$$|H_2 - H_3| \ll_{\pi,k,\ell} \Bigg| \sum_{\substack{|\gamma_\pi| \leqslant T \\ |\beta_\pi - \frac{1}{2}| < \delta}} \left(\left(\frac{Q}{P}\right)^{\beta_\pi - \frac{1}{2}} - 1\right) \Bigg| + \frac{T}{(\log T)^{100}} \ll_{\pi,k,\ell} \frac{T \log^2 V}{\log T}. \tag{5.6}$$

Combining (5.5) and (5.6) and applying triangle inequality, we arrive at our desired conclusion. □

Fix $\xi > 1$. Suppose $T \geqslant 3$ and let α, X, V satisfy

$$0 < \alpha < \frac{1}{m}, \quad X = (\log T)^{\alpha m} \quad \text{and} \quad V = X^{\xi}. \tag{5.7}$$

Equipped with Proposition 5.1, we establish an analogue of Lemma 3.1 for small ranges of X compared to T. Our proof strategy is similar to Lemma 3.1. We highlight only the differences.

Lemma 5.2. *Let $m \in \mathbb{N}$ and $\pi \in \mathscr{A}_m$. Assume Hypothesis $\mathbf{H}_\pi$ and Hypothesis $\mathbf{Z}_\pi$ for $L(s,\pi)$. Suppose k, ℓ be non-negative integers with $k+\ell = 2d$ for some $d \in \mathbb{N}$. Let $T \geqslant 3$ and α, ξ, X, V be as in* (5.7) *satisfying $d\alpha m < 2$. Let $\{a_n\}$ be the sequence defined by* (2.26)*. Then*

$$\frac{1}{N_\pi(T)} \sum_{|\gamma_\pi| \leqslant T} \Big(\sum_{p \leqslant V} \frac{a_p \Lambda_\pi(p)}{p^{\rho_\pi}} \Big)^k \Big(\sum_{q \leqslant V} \frac{a_q \overline{\Lambda_\pi(q)}}{q^{\overline{\rho_\pi}}} \Big)^\ell = \boldsymbol{\delta}_{k\ell} d! \Big(\sum_{p \leqslant V} \frac{a_p^2 |\Lambda_\pi(p)|^2}{p} \Big)^d + O_{\pi,d,\alpha,\xi}(\boldsymbol{\delta}_{k\ell} \max\{1, (\log\log T)^{d-2}\} + (\log T)^{-\varepsilon}),$$

for some $\varepsilon > 0$ depending at most on π, d, α and ξ. Here p, q run over primes up to V and $\boldsymbol{\delta}_{k\ell} = 1$ if $k = \ell$, and zero otherwise.

Proof. All implied constants in the proof are allowed to depend on π, d, α and ξ. For notational simplicity, let

$$\mathcal{B}_{k,\ell} = \frac{1}{N_\pi(T)} \sum_{|\gamma_\pi| \leqslant T} \Big(\sum_{p \leqslant V} \frac{a_p \Lambda_\pi(p)}{p^{\rho_\pi}} \Big)^k \Big(\sum_{q \leqslant V} \frac{a_q \overline{\Lambda_\pi(q)}}{q^{\overline{\rho_\pi}}} \Big)^\ell. \tag{5.8}$$

We break our proof into two cases.

Case 1 : When $k \neq \ell$. Assume $k > \ell$. When $\ell = 0$, similar to Lemma 3.1 we obtain $\mathcal{B}_{k,0} \ll_{\pi,d,\alpha,\xi} (\log T)^{-1}$. Now suppose $\ell > 0$. By (2.2), we can write

$$\begin{aligned}\mathcal{B}_{k,\ell} &= \frac{1}{N_\pi(T)} \sum_{\substack{p_1,p_2,\ldots,p_k \leqslant V \\ q_1,q_2,\ldots,q_\ell \leqslant V}} \Big(\prod_{i=1}^k a_{p_i} \Lambda_\pi(p_i) \Big) \Big(\prod_{j=1}^\ell a_{q_j} \overline{\Lambda_\pi(q_j)} \Big) \sum_{|\gamma_\pi| \leqslant T} P^{-\rho_\pi} Q^{-\overline{\rho_\pi}}, \\ &= \frac{1}{N_\pi(T)} \sum_{\substack{p_1,p_2,\ldots,p_k \leqslant V \\ q_1,q_2,\ldots,q_\ell \leqslant V}} \frac{(\prod_{i=1}^k a_{p_i} \Lambda_\pi(p_i))(\prod_{j=1}^\ell a_{q_j} \overline{\Lambda_\pi(q_j)})}{(PQ)^{\frac{1}{2}}} \sum_{|\gamma_\pi| \leqslant T} (PQ)^{\beta_\pi - \frac{1}{2}} \Big(\frac{Q}{P} \Big)^{i\gamma_\pi},\end{aligned} \tag{5.9}$$

where $P = p_1 p_2 \cdots p_k$, $Q = q_1 q_2 \cdots q_\ell$ and we use the fact that if $L(\rho_\pi, \pi) = 0$ then $L(1-\overline{\rho_\pi}, \pi) = 0$. By Proposition 5.1, the inner sum over γ_π in (5.9) by

$$\sum_{|\gamma_\pi| \leqslant T} \Big(\frac{Q}{P} \Big)^{\rho_\pi - \frac{1}{2}} + O_{\pi,d,\alpha,\xi}\Big(\frac{T \log^2 V}{\log T} \Big). \tag{5.10}$$

By Cauchy–Schwarz and Lemma 2.10, the contribution to (5.9) from the error term in (5.10) is

$$\ll_{\pi,d,\alpha,\xi} \Big(\frac{\log V}{\log T} \Big)^2 \Big(\sum_{p \leqslant V} \frac{a_p |\Lambda_\pi(p)|}{\sqrt{p}} \Big)^{2d} \ll_{\pi,d,\alpha,\xi,\varepsilon} \frac{X^{d+\varepsilon}}{(\log T)^2} \ll_{\pi,d,\alpha,\xi,\varepsilon} (\log T)^{-\varepsilon},$$

for some $\varepsilon > 0$ depending on π, d, α and ξ. To estimate the contribution from the first term in (5.10), we note that Q can never be equal to P. We break into cases depending on whether $P > Q$ or $P < Q$. After applying Corollary 2.6 and 2.7, the overall contribution in Case 1 is $\ll_{\pi,d,\alpha,\xi} (\log T)^{-\varepsilon}$ for some $\varepsilon > 0$ depending on π, d, α and ξ. A concomitant argument follows when $\ell > k$.

Case 2 : When $k = \ell$. We write

$$\mathcal{B}_{d,d} = \frac{1}{N_\pi(T)} \sum_{\substack{p_1,p_2,\ldots,p_d \leqslant V \\ q_1,q_2,\ldots,q_d \leqslant V}} \frac{(\prod_{i=1}^d a_{p_i}\Lambda_\pi(p_i))(\prod_{j=1}^d a_{q_j}\overline{\Lambda_\pi(q_j)})}{(PQ)^{\frac12}} \sum_{|\gamma_\pi|\leqslant T} (PQ)^{\beta_\pi - \frac12}\left(\frac{Q}{P}\right)^{i\gamma_\pi},$$

where $P = p_1p_2\cdots p_d$ and $Q = q_1q_2\cdots q_d$. The main terms again occur when $P = Q$. An application of Proposition 5.1 shows that the contribution from the case $P = Q$ is

$$\sum_{\substack{p_1,p_2,\ldots,p_d \leqslant V \\ q_1,q_2,\ldots,q_d \leqslant V \\ P=Q}} \frac{(\prod_{i=1}^d a_{p_i}\Lambda_\pi(p_i))(\prod_{j=1}^d a_{q_j}\overline{\Lambda_\pi(q_j)})}{P} + O_{\pi,d,\alpha,\xi}((\log T)^{-\varepsilon}). \tag{5.11}$$

Similar to the proof of Lemma 3.1, the first sum in (5.11) is equal to

$$d!\left(\sum_{p\leqslant V}\frac{a_p^2|\Lambda_\pi(p)|^2}{p}\right)^d + O_{\pi,d,\alpha,\xi}(\max\{1,(\log\log T)^{d-2}\}).$$

The cases $P \neq Q$ can be treated as in Case 1. Putting together Cases 1 and 2, the proof follows. □

Lemma 5.3. *Let $m \in \mathbb{N}$ and $\pi \in \mathscr{A}_m$. Assume Hypothesis $\mathbf{H}_\pi$ and Hypothesis $\mathbf{Z}_\pi$ for $L(s,\pi)$. Let $d \in \mathbb{N}$, $T \geqslant 2$ and α, ξ, X, V be as in* (5.7) *satisfying $d\alpha m < 2$. Let $\{a_n\}$ be the sequence defined by* (2.26)*. Then*

$$\frac{1}{N_\pi(T)}\sum_{|\gamma_\pi|\leqslant T}\left(\operatorname{Re}\sum_{p\leqslant V}\frac{a_p\Lambda_\pi(p)}{p^{\rho_\pi}}\right)^{2d} = \boldsymbol{\mu}_{2d}\left(\frac12\sum_{p\leqslant V}\frac{a_p^2|\Lambda_\pi(p)|^2}{p}\right)^d + O_{\pi,d,\alpha,\xi}(\max\{1,(\log\log T)^{d-2}\}),$$

where $\boldsymbol{\mu}_{2d}$ is given by (1.19)*. Also, if $(2d-1)\alpha m < 4$ then for some $\varepsilon = \varepsilon(\pi,d,\alpha,\xi) > 0$, we have*

$$\frac{1}{N_\pi(T)}\sum_{|\gamma_\pi|\leqslant T}\left(\operatorname{Re}\sum_{p\leqslant V}\frac{a_p\Lambda_\pi(p)}{p^{\rho_\pi}}\right)^{2d-1} \ll_{\pi,d,\alpha,\xi} (\log T)^{-\varepsilon}.$$

Proof. The proof precisely follows the arguments in Lemma 3.2. □

Recall in (3.10) and (3.11), we defined

$$\operatorname{Av}_\pi(T,X,V,\{a_n\}) = \frac{1}{N_\pi(T)}\sum_{|\gamma_\pi|\leqslant T}\operatorname{Re}\sum_{n\leqslant V}\frac{a_n\Lambda_\pi(n)}{n^{\rho_\pi}}$$

$$\text{and}\quad \operatorname{Var}_\pi(T,X,V,\{a_n\}) = \frac12\sum_{n\leqslant V}\frac{a_n^2|\Lambda_\pi(n)|^2}{n} - \operatorname{Av}_\pi^2(T,X,V,\{a_n\}).$$

The key difference in our work here compared to Section 3 is the range of X. However this doesn't change the structure of the asymptotics for Av_π or Var_π as in Lemma 3.3. In particular, following the proof of Lemma 3.3 and using the estimates from Lemma 2.10, we have for $T \geqslant 2$ and α, ξ, X, V be as in (5.7) satisfying $\alpha m < 4$,

$$\operatorname{Av}_\pi(T,X,V,\{a_n\}) = -\frac{\alpha\log\log T}{\log T} + O_{\pi,\alpha,\xi}\left(\frac{1}{\log T}\right), \tag{5.12}$$

$$\text{and}\quad \operatorname{Var}_\pi(T,X,V,\{a_n\}) = \frac{\alpha m\log\log T}{4} + O_{\pi,\alpha,\xi}(1). \tag{5.13}$$

We are now ready to estimate our moments involving sums over prime powers.

Lemma 5.4. *Let $m \in \mathbb{N}$ and $\pi \in \mathscr{A}_m$. Assume Hypothesis $\mathbf{H}_\pi$ and Hypothesis $\mathbf{Z}_\pi$ for $L(s,\pi)$. Let $d \in \mathbb{N}$, $T \geqslant 2$ and α, ξ, X, V be as in* (5.7) *satisfying $d\alpha m < 2$. Let $\{a_n\}$ be the sequence defined by* (2.26)*. Then*

$$\frac{1}{N_\pi(T)} \sum_{|\gamma_\pi| \leqslant T} \left(\operatorname{Re} \sum_{n \leqslant V} \frac{a_n \Lambda_\pi(n)}{n^{\rho_\pi}} \right)^{2d} = \boldsymbol{\mu}_{2d} \mathrm{Var}_\pi^d + O_{\pi,d,\alpha,\xi}((\log\log T)^{d-\frac{1}{2}}),$$

where $\boldsymbol{\mu}_{2d}$ is given by (1.19) *and* Var_π *is defined by* (3.11)*. Also, if $(2d-1)\alpha m < 4$ then*

$$\frac{1}{N_\pi(T)} \sum_{|\gamma_\pi| \leqslant T} \left(\operatorname{Re} \sum_{n \leqslant V} \frac{a_n \Lambda_\pi(n)}{n^{\rho_\pi}} \right)^{2d-1} \ll_{\pi,d,\alpha,\xi} 1.$$

Proof. The proof is similar to that of Lemma 3.4 with slight adjustments. For even moments, we write

$$\sum_{|\gamma_\pi| \leqslant T} \left(\operatorname{Re} \sum_{n \leqslant V} \frac{a_n \Lambda_\pi(n)}{n^{\rho_\pi}} \right)^{2d} = \sum_{j=0}^{2d} \binom{2d}{j} \sum_{|\gamma_\pi| \leqslant T} \left(\operatorname{Re} \sum_{p \leqslant V} \frac{a_p \Lambda_\pi(p)}{p^{\rho_\pi}} \right)^{j} \left(\operatorname{Re} \sum_{\substack{n \leqslant V \\ n=p^k, k \geqslant 2}} \frac{a_n \Lambda_\pi(n)}{n^{\rho_\pi}} \right)^{2d-j}.$$

By Lemma 5.3 and (5.13), when $j = 2d$, we have

$$\frac{1}{N_\pi(T)} \sum_{|\gamma_\pi| \leqslant T} \left(\operatorname{Re} \sum_{p \leqslant V} \frac{a_p \Lambda_\pi(p)}{p^{\rho_\pi}} \right)^{2d} = \boldsymbol{\mu}_{2d} \mathrm{Var}_\pi^d + O_{\pi,d,\alpha,\xi}((\log\log T)^{d-1}), \tag{5.14}$$

For $j = 0$, we break the sum over ρ_π into two cases, $\beta_\pi > \frac{1}{4} + \frac{\theta_m}{2}$ and otherwise. Note that

$$\Bigg| \sum_{\substack{n \leqslant V \\ n=p^k, k \geqslant 2}} \frac{a_n \Lambda_\pi(n)}{n^{\rho_\pi}} \Bigg|^{2d} \ll_d \Bigg| \sum_{\substack{n=p^k \leqslant V \\ 2 \leqslant k \leqslant 2(m^2+1)}} \frac{a_{p^2} \Lambda_\pi(p^2)}{p^{2\rho_\pi}} \Bigg|^{2d} + \Bigg| \sum_{\substack{n=p^k \leqslant V \\ k > 2(m^2+1)}} \frac{a_n \Lambda_\pi(n)}{n^{\rho_\pi}} \Bigg|^{2d}.$$

When $\beta_\pi > \frac{1}{4} + \frac{\theta_m}{2}$, we have

$$\sum_{\substack{n=p^k \leqslant V \\ k > 2(m^2+1)}} \frac{a_n \Lambda_\pi(n)}{n^{\rho_\pi}} \ll_{\pi,d,\alpha,\xi} 1,$$

which implies that

$$\frac{1}{N_\pi(T)} \sum_{\substack{|\gamma_\pi| \leqslant T \\ \beta_\pi > \frac{1}{4} + \frac{\theta_m}{2}}} \Bigg| \sum_{\substack{n=p^k \leqslant V \\ k > 2(m^2+1)}} \frac{a_n \Lambda_\pi(n)}{n^{\rho_\pi}} \Bigg|^{2d} \ll_{\pi,d,\alpha,\xi} 1. \tag{5.15}$$

On the other hand, for $\beta_\pi \leqslant \frac{1}{4} + \frac{\theta_m}{2}$, using Hypothesis $\mathbf{Z}_\pi$,

$$\frac{1}{N_\pi(T)} \sum_{\substack{|\gamma_\pi| \leqslant T \\ \beta_\pi \leqslant \frac{1}{4} + \frac{\theta_m}{2}}} \Bigg| \sum_{\substack{n=p^k \leqslant V \\ k > 2(m^2+1)}} \frac{a_n \Lambda_\pi(n)}{n^{\rho_\pi}} \Bigg|^{2d} \ll_{\pi,d,\alpha,\xi} 1. \tag{5.16}$$

Therefore putting together (5.15) and (5.16), the contribution from $j = 0$ is

$$\frac{1}{N_\pi(T)} \sum_{|\gamma_\pi| \leqslant T} \left(\operatorname{Re} \sum_{\substack{n \leqslant V \\ n=p^k, k \geqslant 2}} \frac{a_n \Lambda_\pi(n)}{n^{\rho_\pi}} \right)^{2d} \ll_{\pi,d,\alpha,\xi} \frac{1}{N_\pi(T)} \sum_{|\gamma_\pi| \leqslant T} \Bigg| \sum_{\substack{n=p^k \leqslant V \\ 2 \leqslant k \leqslant 2(m^2+1)}} \frac{a_{p^k} \Lambda_\pi(p^k)}{p^{k\rho_\pi}} \Bigg|^{2d} + 1.$$

From this point, we can proceed using the same approach as in Lemma 3.4. We begin by applying Jensen's inequality, and then handle each case for k individually, following the reasoning outlined in Lemma 5.2. In particular, we obtain

$$\frac{1}{N_\pi(T)}\sum_{|\gamma_\pi|\leqslant T}\Bigg|\sum_{\substack{n=p^k\leqslant V\\ 2\leqslant k\leqslant 2(m^2+1)}}\frac{a_{p^k}\Lambda_\pi(p^k)}{p^{k\rho_\pi}}\Bigg|^{2d}\ll_{\pi,d,\alpha,\xi} 1,$$

which implies the contribution when $j=0$ is $O_{\pi,d,\alpha,\xi}(1)$. Finally when $j\neq 0,2d$, by Hölder's inequality, the sum corresponding to each such j is $\ll_{\pi,d,\alpha,\xi} (\log\log T)^{d-\frac{1}{2}}$. Combining the estimates for different values of j, we arrive at our desired result. In the case of odd moments, the argument is precisely as in Lemma 3.4 with necessary applications of Lemma 5.3. □

6. Proofs of Theorems 1.11 and 1.12

Let $m\in\mathbb{N}$ and $\pi\in\mathscr{A}_m$. Suppose $T\geqslant 3$ and α,ξ,X,V be as defined in (5.7) satisfying $\alpha m<4$. Recall from Section 1, the shifted sums

$$\widetilde{\mathcal{S}}_\pi(X,T,\rho')=\mathcal{S}_\pi(X,T,\rho')-X^{-2}\frac{L'}{L}(\rho'_\pi-2,\pi),$$

$$\text{and}\quad \operatorname{Re}\widetilde{\mathcal{S}}_{\mathrm{av},\pi}(X,T)=\frac{1}{N_\pi(T)}\sum_{|\gamma'_\pi|\leqslant T}\operatorname{Re}\widetilde{\mathcal{S}}_\pi(X,T,\rho'_\pi).$$

Following the proofs of Lemmata 2.2 and 2.4, for all but $O_\pi(\sqrt{T}\log T)$ zeros $\rho'=\beta'_\pi+i\gamma'_\pi$ of $L(s,\pi)$ with $0<|\gamma'_\pi|\leqslant T$, we have

$$\widetilde{\mathcal{S}}_\pi(X,T,\rho'_\pi)=-\sum_{n=1}^{\infty}\frac{a_n\Lambda_\pi(n)}{n^{\rho'_\pi}}+O_\pi\bigg(\frac{X\log T}{\sqrt{T}}\bigg).\tag{6.1}$$

Again, similar to Section 4, we slightly adjust (5.7) by allowing $\xi>3$. We prove the following lemma.

Lemma 6.1. *Let $m\in\mathbb{N}$ and $\pi\in\mathscr{A}_m$. Assume Hypothesis $\mathbf{H}_\pi$ and Hypothesis $\mathbf{Z}_\pi$ for $L(s,\pi)$. Let $T\geqslant 3$ and α,ξ,X,V be as in* (5.7) *satisfying $\alpha m<4$ and $\xi>3$. Let $\{a_n\}$ be the sequence in* (2.26). *Then*

$$\operatorname{Re}\widetilde{\mathcal{S}}_{\mathrm{av},\pi}(X,T)=\frac{\alpha\log\log T}{\log T}+O_{\pi,\alpha,\xi}\bigg(\frac{1}{\log T}\bigg).$$

Proof. We have

$$\begin{aligned}\operatorname{Re}\widetilde{\mathcal{S}}_{\mathrm{av},\pi}(X,T)&=-\frac{1}{N_\pi(T)}\sum_{|\gamma'_\pi|\leqslant T}\operatorname{Re}\sum_{n=1}^{\infty}\frac{a_n\Lambda_\pi(n)}{n^{\rho'_\pi}}\\&\quad+\frac{1}{N_\pi(T)}\sum_{\gamma'_\pi\in\mathcal{E}}\bigg(\operatorname{Re}\widetilde{\mathcal{S}}_\pi(X,T,\rho'_\pi)+\operatorname{Re}\sum_{n=1}^{\infty}\frac{a_n\Lambda_\pi(n)}{n^{\rho'_\pi}}\bigg)+O_\pi\bigg(\frac{X\log T}{\sqrt{T}}\bigg),\end{aligned}\tag{6.2}$$

where $\mathcal{E}$ is an exceptional set of non-trivial zeros $\rho'_\pi=\beta_\pi+i\gamma'_\pi$ of $L(s,\pi)$ with $|\gamma'_\pi|\leqslant T$ and $\#\mathcal{E}\ll_\pi\sqrt{T}\log T$. The sum over the exceptional zeros is

$$\ll_\pi \frac{1}{N_\pi(T)}\sum_{\gamma'\in\mathcal{E}}(X\log T+X\log X)\ll_\pi\frac{X\log T}{\sqrt{T}}.\tag{6.3}$$

Similar to Lemma 4.1, by using a dyadic decomposition argument, we have

$$\frac{1}{N_\pi(T)}\sum_{|\gamma'_\pi|\leqslant T}\operatorname{Re}\sum_{n=1}^{\infty}\frac{a_n\Lambda_\pi(n)}{n^{\rho'_\pi}}-\mathrm{Av}_\pi=\frac{1}{N_\pi(T)}\sum_{|\gamma'_\pi|\leqslant T}\operatorname{Re}\sum_{n>V}\frac{a_n\Lambda_\pi(n)}{n^{\rho'_\pi}}\ll_{\pi,\alpha,\xi}(\log T)^{-1}.\tag{6.4}$$

Putting together (6.3) and (6.4) into (6.2) and using (5.12), the proof follows. □

Proof of Theorem 1.12. First assume the exponent is even and write $r = 2d$. We let $X = (\log T)^{\alpha m}$ with $d\alpha m < 2$. Choose $V = X^{\xi}$ for some absolute constant $\xi > 3$. Lemma 2.4, together with the argument we used to establish (6.4), shows that for all non-trivial zeros $\rho'_\pi = \beta'_\pi + i\gamma'_\pi$ of $L(s,\pi)$ with $|\gamma'_\pi| \leqslant T$ outside an exceptional set $\mathcal{E}$ of size at most $O_\pi(\sqrt{T}\log T)$, we have

$$\operatorname{Re}\widetilde{\mathcal{S}}_\pi(X,T,\rho'_\pi) = -\operatorname{Re}\sum_{n\leqslant V}\frac{a_n\Lambda_\pi(n)}{n^{\rho'_\pi}} + \widetilde{E}_\pi(X,T,\rho_\pi),$$

where $\widetilde{E}_\pi(X,T,\rho_\pi) \ll_{\pi,\alpha} X^{-1}$. Then we can write

$$\begin{aligned}\frac{1}{N_\pi(T)}&\sum_{|\gamma'_\pi|\leqslant T}(\operatorname{Re}\widetilde{\mathcal{S}}_\pi(X,T,\rho'_\pi))^{2d}\\ &= \frac{1}{N_\pi(T)}\sum_{|\gamma'_\pi|\leqslant T}\left(-\operatorname{Re}\sum_{n\leqslant V}\frac{a_n\Lambda_\pi(n)}{n^{\rho'_\pi}} + \widetilde{E}_\pi(X,T,\rho_\pi)\right)^{2d}\\ &\quad+\frac{1}{N_\pi(T)}\sum_{\gamma'\in\mathcal{E}}\left((\operatorname{Re}\widetilde{\mathcal{S}}_\pi(X,T,\rho'_\pi))^{2d} - \left(-\operatorname{Re}\sum_{n\leqslant V}\frac{a_n\Lambda_\pi(n)}{n^{\rho'_\pi}} + \widetilde{E}_\pi(X,T,\rho_\pi)\right)^{2d}\right) = \widetilde{\mathcal{U}}_1 + \widetilde{\mathcal{U}}_2,\end{aligned}$$

say. An application of Lemma 5.4 shows that

$$\widetilde{\mathcal{U}}_1 = \boldsymbol{\mu}_{2d}\mathrm{Var}_\pi^d + O_{\pi,d,\alpha}((\log\log T)^{d-\frac{1}{2}}).$$

An argument similar to how we proved (6.3) shows that $\widetilde{\mathcal{U}}_2 \ll_{\pi,d,\alpha} T^{-\frac{1}{2}+\varepsilon}$ for any $\varepsilon = \varepsilon(\pi,d,\alpha) > 0$. Hence we obtain

$$\frac{1}{N_\pi(T)}\sum_{|\gamma'_\pi|\leqslant T}(\operatorname{Re}\widetilde{\mathcal{S}}_\pi(X,T,\rho'_\pi))^{2d} = \boldsymbol{\mu}_{2d}\mathrm{Var}_\pi^d + O_{\pi,d,\alpha}((\log\log T)^{d-\frac{1}{2}}). \tag{6.5}$$

Now applying (5.13) and Lemma 6.1, it follows that

$$\widetilde{\mathcal{M}}_{\pi,2d}((\log T)^{\alpha m},T) = \boldsymbol{\mu}_{2d}\left(\frac{\alpha m\log\log T}{4}\right)^d + O_{\pi,d,\alpha}((\log\log T)^{d-\frac{1}{2}}).$$

When the exponent is odd, the proof is similar except that $\mathcal{U}_1 \ll_{\pi,d,\alpha} 1$. □

Proof of Theorem 1.11. The proof follows precisely along the lines of the proof of Theorem 1.4. □

7. General weight functions: Proofs of Theorems 1.14 and 1.15

7.1. Preliminaries.

Recall from Section 1.4, for $X \geqslant 2$ and $\Psi \in C_c^\infty(0,\infty)$ a fixed non-negative compactly supported smooth function, we defined

$$\Psi_X(x) := \Psi\left(\frac{x}{X}\right), \quad x\in\mathbb{R}$$

Fix $\xi > 1$. Suppose $T \geqslant 2$ and let α, X, V satisfy

$$0 < \alpha < \frac{1}{m}, \quad X = T^{\alpha m} \quad \text{and} \quad V = X^{\xi}. \tag{7.1}$$

We first record the following estimates.

Lemma 7.1. *Let $m\in\mathbb{N}$, $\pi\in\mathscr{A}_m$, and assume Hypothesis $\mathbf{H}_\pi$ for $L(s,\pi)$. Let $\Psi\in C_c^\infty(0,\infty)$ be a fixed non-negative compactly supported smooth function. Let $T\geqslant 3$ and α,ξ,X,V be as in* (7.1). *Then*

$$\sum_{n\leqslant V}\frac{\Psi_X(n)|\Lambda_\pi(n)|^2}{n} = \alpha m\log T\int_0^\infty\frac{\Psi(t)}{t}\,\mathrm{d}t + O_{\pi,\Psi,\alpha,\xi}(1) \tag{7.2}$$

and

$$\sum_{n\leqslant V}\frac{\Psi_X^2(n)|\Lambda_\pi(n)|^2}{n}=\alpha m\log T\int_0^\infty\frac{\Psi^2(t)}{t}\,\mathrm{d}t+O_{\pi,\Psi,\alpha,\xi}(1). \tag{7.3}$$

Proof. Both (7.2) and (7.3) follow by an application of partial summation along with Lemma 2.9. □

7.2. **Explicit Formula.** As before, we conduct the bulk of our analysis on a sum of primes related by an explicit formula to our sum over zeros. We begin by proving the following lemma.

Lemma 7.2. *Let $m\in\mathbb{N}$ and $\pi\in\mathscr{A}_m$. Assume $\varphi\in C_c^\infty(2,\infty)$. Let $s\in\mathbb{C}$ such that $L(s,\pi)\neq 0$. Then*

$$\sum_{n=1}^\infty\frac{\Lambda_\pi(n)\varphi(n)}{n^s}=\delta_\pi\hat\varphi(1-s)-\sum_{\rho_\pi}\hat\varphi(\rho_\pi-s)-\sum_{\substack{k\in\mathbb{N}\cup\{0\}\\1\leqslant j\leqslant m}}\sum\hat\varphi(-2k-\kappa_\pi(j)-s), \tag{7.4}$$

where $\kappa_\pi(j)\in\mathbb{C}$ are the spectral parameters of π. When $s=1$, the first term in the right hand side of (7.4) *should be interpreted as $\delta_\pi\hat\varphi(0)$, where $\delta_\pi=1$ if $L(s,\pi)=\zeta(s)$ and zero otherwise.*

Proof. Assume $s\neq 1$. For $z\neq 0$, by integration by parts,

$$\hat\varphi(z)=\int_2^\infty\varphi(y)y^{z-1}\,\mathrm{d}y=-\frac{1}{z}\int_2^\infty\varphi'(y)y^z\,\mathrm{d}y. \tag{7.5}$$

We apply the fundamental theorem of calculus to find

$$\sum_{n=1}^\infty\frac{\Lambda_\pi(n)\varphi(n)}{n^s}=\sum_{n\geqslant 2}\frac{\Lambda_\pi(n)}{n^s}\int_n^\infty-\varphi'(y)\,\mathrm{d}y=-\int_2^\infty\varphi'(y)\sum_{2\leqslant n\leqslant y}\frac{\Lambda_\pi(n)}{n^s}\,\mathrm{d}y.$$

From (2.11) of Lemma 2.1, we see

$$\begin{aligned}\sum_{n=1}^\infty\frac{\Lambda_\pi(n)\varphi(n)}{n^s}&=\int_2^\infty\varphi'(y)\frac{L'}{L}(s,\pi)\,\mathrm{d}y+\frac{\delta_\pi}{1-s}\int_2^\infty\varphi'(y)y^{1-s}\,\mathrm{d}y+\int_2^\infty\sum_{\rho_\pi}\frac{\varphi'(y)y^{\rho_\pi-s}}{\rho_\pi-s}\,\mathrm{d}y\\&\quad-\int_2^\infty\sum_{\substack{k\in\mathbb{N}\cup\{0\}\\1\leqslant j\leqslant m}}\sum\frac{\varphi'(y)y^{-2k-\kappa_\pi(j)-s}}{2k+\kappa_\pi(j)+s}.\end{aligned} \tag{7.6}$$

Note we may apply (2.11) of Lemma 2.1 because $\mathbb{N}$ has Lebesgue measure zero. We have that the first integral of (7.6) is zero by the Fundamental Theorem of Calculus. If the representation is trivial and $s\neq 1$, we apply (7.5) to find that the second term of the right-hand side of (7.6) equals $\hat\varphi(1-s)$. We now proceed to integral of the sum over the zeros of $L(s,\pi)$. Note, we have for $\mathrm{Re}(z)\leqslant 1$,

$$\hat\varphi(z)\ll_\varphi 2^{\mathrm{Re}(z)-1}. \tag{7.7}$$

Furthermore, when $z\neq 0,-1$ with $\mathrm{Re}\,z\leqslant 1$, we find

$$\hat\varphi(z)\ll_\varphi\frac{1}{|z||z+1|}. \tag{7.8}$$

Using (7.5) with (7.7) and (7.8), we may apply Fubini's theorem and interchange the sums with our integrals in (7.6). With this, we arrive at (7.4) for $s\neq 1$. When $s=1$ and π is the trivial representation, we exchange the first two terms of the right-hand side of (7.6) for $\hat\varphi(0)$; the rest of the work is left unchanged. □

We now extend Lemma 7.2 to include $s\in\mathbb{C}$ that are nontrivial zeros of $L(s,\pi)$.

Lemma 7.3. *Let $m \in \mathbb{N}$ and $\pi \in \mathscr{A}_m$. Assume $\varphi \in C_c^\infty(2,\infty)$. Let ρ'_π be a non-trivial zero of $L(s,\pi)$. Then we have*

$$\sum_{n=1}^{\infty} \frac{\Lambda_\pi(n)\varphi(n)}{n^{\rho'_\pi}} = \delta_\pi \hat{\varphi}(1-\rho'_\pi) - \sum_{\rho_\pi} \hat{\varphi}(\rho_\pi - \rho'_\pi) - \sum_{\substack{k \in \mathbb{N}\cup\{0\} \\ 1 \leqslant j \leqslant m}} \sum \hat{\varphi}(-2k - \kappa_\pi(j) - \rho'_\pi). \tag{7.9}$$

where ρ_π run over non-trivial zeros of $L(s,\pi)$ and $\kappa_\pi(j) \in \mathbb{C}$ are the spectral parameters of π.

Proof. Fix ρ'_π to be a zero of $L(s,\pi)$. To prove this lemma, we use the uniqueness principle of complex analysis. As such it requires us to understand where each term of (7.4) is holomorphic. It is clear that the Dirichlet series of (7.4) is entire. We also find that

$$\sum_{\substack{k \in \mathbb{N}\cup\{0\} \\ 1 \leqslant j \leqslant m}} \sum \hat{\varphi}(-2k - \kappa_\pi(j) - s)$$

is analytic in $\operatorname{Re} s \geqslant 0$. This follows from $\operatorname{Re} \kappa(j) \geqslant -\frac{1}{2}$. Indeed, for all $k \in \mathbb{N} \cup \{0\}$,

$$\sum_{1 \leqslant j \leqslant m} \hat{\varphi}(-2k - \kappa_\pi(j) - s) \ll_\pi 2^{-2k} \int_2^\infty t^{-1/2} \varphi(t) dt \ll_{\pi,\varphi} 2^{-2k}.$$

Because of the support of φ, we find that its Mellin transform is entire. Since $\hat{\varphi}(s)$ is entire, we have the sum above is holomorphic. Hence there exists $\varepsilon(\rho'_\pi) > 0$ such that for $s \in \mathbb{C}$ with $|s - \rho'_\pi| \leqslant \varepsilon(\rho'_\pi)$, the sum over the nontrivial zeros is uniformly bounded. We now apply the uniqueness principle of complex analysis on the small ball of radius $\varepsilon(\rho'_\pi)$ centered at ρ'_π and (7.9) follows. ☐

Recall that in Section 1.4, for $X, T \geqslant 2$ and ρ'_π be a nontrivial zero of $L(s,\pi)$, we defined

$$\mathcal{S}_{\pi,\Psi}(X,T,\rho'_\pi) = \sum_{|\operatorname{Im} \rho_\pi| \leqslant T} \hat{\Psi}_X(\rho_\pi - \rho'_\pi).$$

We now view $\mathcal{S}_{\pi,\Psi}(X,T,\rho'_\pi)$ as a sum over prime powers.

Corollary 7.4. *Let $m \in \mathbb{N}$ and $\pi \in \mathscr{A}_m$. Let $\Psi \in C_c^\infty(0,\infty)$ be a fixed non-negative compactly supported smooth function. Let $X, T \geqslant 2$. Then for all nontrivial zeros $\rho'_\pi = \beta'_\pi + i\gamma'_\pi$ of $L(s,\pi)$ with $|\gamma'_\pi| \leqslant T$ outside an exceptional set of size at most $O_\pi(\sqrt{T}\log T)$, we have*

$$\mathcal{S}_{\pi,\Psi}(X,T,\rho'_\pi) = -\sum_{n=1}^{\infty} \frac{\Psi_X(n)\Lambda_\pi(n)}{n^{\rho'_\pi}} + O_{\pi,\Psi}\bigg(\frac{X^{1-\beta'_\pi}}{T} + \frac{X\log T}{\sqrt{T}}\bigg).$$

Moreover, assuming the Riemann Hypothesis for $L(s,\pi)$, we have

$$\mathcal{S}_{\pi,\Psi}(X,T,\rho'_\pi) = -\sum_{n=1}^{\infty} \frac{\Psi_X(n)\Lambda_\pi(n)}{n^{\rho'_\pi}} + O_{\pi,\Psi}\bigg(\frac{X^{\frac{1}{2}}}{T} + \frac{\log T}{\sqrt{T}}\bigg).$$

Proof. Fix a nontrivial zero $\rho'_\pi = \beta'_\pi + i\gamma'_\pi$ of $L(s,\pi)$. As before, we may assume that $\sqrt{T} < |\gamma'_\pi| < T - \sqrt{T}$. Using integration by parts, we derive the following bound

$$\hat{\Psi}_X(z) \ll_\Psi \frac{\max\{X^{\operatorname{Re} z}, X^{-1}\}}{|z||z+1|} \tag{7.10}$$

for $z \neq 0, -1$. We consider each term of the explicit formula of Lemma 7.3, separately. In the case π is the trivial representation, we find by (7.10)

$$\hat{\Psi}_X(1-\rho'_\pi) \ll_\Psi \frac{X^{\operatorname{Re}(1-\rho'_\pi)}}{|1-\rho'_\pi||2-\rho'_\pi|} \ll_\Psi \frac{X^{1-\beta'_\pi}}{T}.$$

to the first, which holds for $s \neq 0$. Next, we truncate the infinite sum over our nontrivial zeros. Note, we have by (7.10)

$$\hat{\Psi}_X(\rho_\pi - \rho'_\pi) \ll_\Psi \frac{X^{\mathrm{Re}(\rho'_\pi - \rho_\pi)}}{|\rho_\pi - \rho'_\pi||\rho_\pi - \rho'_\pi + 1|} \ll_\Psi \frac{X^{\mathrm{Re}(\rho'_\pi - \rho_\pi)}}{|\gamma_\pi - \gamma'_\pi|^2},$$

when $\rho_\pi - \rho'_\pi \neq 0, -1$. By following the proof of (2.34), these terms are

$$\ll_{\pi,\Psi} \frac{X \log T}{\sqrt{T}}.$$

Assuming RH for $L(s,\pi)$, we have $X^{\mathrm{Re}(\rho'_\pi - \rho_\pi)} \ll 1$. Finally, we find that the sum over the trivial zeros is bounded by our error for the tail of our sum over nontrivial zeros by a similar argument using (7.10). □

7.3. **Proofs of Theorems 1.14 and 1.15.** We first note that Theorem 1.15 follows immediately from Theorem 1.14 using arguments similar to the proof of Theorem 1.4. Towards the proof of Theorem 1.14, much of the work below follows from the analysis of Section 3 and Section 4. For the benefit of the reader, we include the statements of the lemmata necessary for the proof of Theorem 1.14. The key difference is the use of Lemma 7.1 instead of Lemma 2.10; the rest is left largely unchanged. We also mention that the condition $\xi > 1$ in (7.1) is sufficient here, unlike the constraint $\xi > \frac{4}{3}$ in Section 4.

Lemma 7.5. *Let $m \in \mathbb{N}, \pi \in \mathscr{A}_m$ and $\theta_m \in [0, \frac{1}{2} - \frac{1}{m^2+1}]$ be an admissible exponent towards the Ramanujan conjecture for $L(s,\pi)$. Assume Hypothesis $\mathbf{H}_\pi$ and the Riemann Hypothesis for $L(s,\pi)$. Let $\Psi \in C_c^\infty(0,\infty)$ be a fixed non-negative compactly supported smooth function. Let k, ℓ be non-negative integers with $k + \ell = 2d$ for some $d \in \mathbb{N}$. Let $T \geqslant 2$ and α, ξ, X, V be as in* (7.1) *satisfying*

$$\alpha < \frac{1}{2dm(1+\xi\theta_m)}.$$

Then

$$\frac{1}{N_\pi(T)} \sum_{|\gamma_\pi| \leqslant T} \Big(\sum_{p \leqslant V} \frac{\Psi_X(p)\Lambda_\pi(p)}{p^{\frac{1}{2}+i\gamma_\pi}} \Big)^k \Big(\sum_{q \leqslant V} \frac{\Psi_X(p)\overline{\Lambda_\pi(q)}}{q^{\frac{1}{2}-i\gamma_\pi}} \Big)^\ell = \delta_{k\ell} d! \Big(\sum_{p \leqslant V} \frac{\Psi_X^2(p)|\Lambda_\pi(p)|^2}{p} \Big)^d + O_{\pi,\Psi,d,\alpha,\xi}(\max\{1, (\log X)^{d-2}\})$$

where p and q run over the primes up to V and $\delta_{k\ell} = 1$ if $k = \ell$ and zero otherwise.

Lemma 7.6. *Let $m \in \mathbb{N}, \pi \in \mathscr{A}_m$ and $\theta_m \in [0, \frac{1}{2} - \frac{1}{m^2+1}]$ be an admissible exponent towards the Ramanujan conjecture for $L(s,\pi)$. Assume the Riemann Hypothesis and Hypothesis $\mathbf{H}_\pi$ for $L(s,\pi)$. Let $\Psi \in C_c^\infty(0,\infty)$ be a fixed non-negative compactly supported smooth function. Let $d \in \mathbb{N}$, $T \geqslant 2$ and α, ξ, X, V be as in* (7.1) *satisfying*

$$\alpha < \frac{1}{2dm(1+\xi\theta_m)}.$$

Then

$$\frac{1}{N_\pi(T)} \sum_{|\gamma_\pi| \leqslant T} \Big(\mathrm{Re} \sum_{p \leqslant V} \frac{\Psi_X(p)\Lambda_\pi(p)}{p^{\frac{1}{2}+i\gamma_\pi}} \Big)^{2d} = \boldsymbol{\mu}_{2d} \Big(\frac{1}{2} \sum_{p \leqslant V} \frac{\Psi_X^2(p)|\Lambda_\pi(p)|^2}{p} \Big)^d + O_{\pi,\Psi,d,\alpha,\xi}(\max\{1, (\log T)^{d-2}\}).$$

where $\boldsymbol{\mu}_{2d}$ is defined by (1.19). *Also, if*

$$\alpha < \frac{1}{m(2d-1)(1+\xi\theta_m)},$$

then we have

$$\frac{1}{N_\pi(T)} \sum_{|\gamma_\pi| \leqslant T} \Big(\mathrm{Re} \sum_{p \leqslant X} \frac{\Psi_X(p)\Lambda_\pi(p)}{p^{\frac{1}{2}+i\gamma_\pi}} \Big)^{2d-1} \ll_{\pi,\Psi,d,\alpha,\xi} (\log T)^{d-1}.$$

Lemma 7.7. *Let $m \in \mathbb{N}, \pi \in \mathscr{A}_m$ and $\theta_m \in [0, \frac{1}{2} - \frac{1}{m^2+1}]$ be an admissible exponent towards the Ramanujan conjecture for $L(s,\pi)$. Assume the Riemann Hypothesis and Hypothesis $\mathbf{H}_\pi$ for $L(s,\pi)$. Let $\Psi \in C_c^\infty(0,\infty)$ be a fixed non-negative compactly supported smooth function. Let $d \in \mathbb{N}$, $T \geqslant 2$ and α, ξ, X, V be as in* (7.1) *satisfying*

$$\alpha < \frac{1}{2dm(1+\xi\theta_m)}.$$

Then

$$\frac{1}{N_\pi(T)} \sum_{|\gamma_\pi| \leqslant T} \left(\operatorname{Re} \sum_{n \leqslant V} \frac{\Psi_X(n)\Lambda_\pi(n)}{n^{\frac{1}{2}+i\gamma_\pi}} \right)^{2d} = \boldsymbol{\mu}_{2d} \Big(\frac{1}{2} \sum_{n \leqslant V} \frac{\Psi_X(n)|\Lambda_\pi(n)|^2}{n} \Big)^d + O_{\pi,\Psi,d,\alpha,\xi}\big((\log X)^{d-\frac{1}{2}}\big)$$

where $\boldsymbol{\mu}_{2d}$ is defined by (1.19)*. Also, if*

$$\alpha < \frac{1}{m(2d-1)(1+\xi\theta_m)},$$

then we have

$$\frac{1}{N_\pi(T)} \sum_{|\gamma_\pi| \leqslant T} \left(\operatorname{Re} \sum_{n \leqslant V} \frac{\Psi_X(n)\Lambda_\pi(n)}{n^{\frac{1}{2}+i\gamma_\pi}} \right)^{2d-1} \ll_{\pi,\Psi,d,\alpha,\xi} (\log X)^{d-1}.$$

Our final lemma establishes the average of $\mathcal{S}_{\pi,\Psi}(X,T,\rho'_\pi)$ as ρ'_π vary over the zeros of $L(s,\pi)$. For $X, T \geqslant 2$, we set

$$\operatorname{Re} \mathcal{S}_{\text{av},\pi,\Psi}(X,T) := \frac{1}{N_\pi(T)} \sum_{|\operatorname{Im} \rho'_\pi| \leqslant T} \operatorname{Re} \mathcal{S}_{\pi,\Psi}(X,T,\rho'_\pi). \tag{7.11}$$

Lemma 7.8. *Let $m \in \mathbb{N}, \pi \in \mathscr{A}_m$ and $\theta_m \in [0, \frac{1}{2} - \frac{1}{m^2+1}]$ be an admissible exponent towards the Ramanujan conjecture for $L(s,\pi)$. Assume Hypothesis $\mathbf{H}_\pi$ and the Riemann Hypothesis for $L(s,\pi)$. Let $\Psi \in C_c^\infty(0,\infty)$ be a fixed non-negative compactly supported smooth function. Let $T \geqslant 2$ and α, ξ, X, V be as in* (7.1) *satisfying*

$$\alpha < \frac{1}{m(1+\xi\theta_m)}.$$

Then

$$\operatorname{Re} \mathcal{S}_{\text{av},\pi,\Psi}(X,T) = -\alpha \int_0^\infty \frac{\Psi(t)}{t}\, \mathrm{d}t + O_{\pi,\Psi,\alpha,\xi}\Big(\frac{1}{\log T}\Big).$$

From here on, the techniques from Section 4 can be adapted to complete the proof of Theorem 1.14.

8. Triple Correlation : Proof of Theorem 1.16

8.1. **Sums over Zeros to Sums over Primes.** Recall from Section 1.4 that for $X_1, X_2, T \geqslant 2$ and ρ_π a nontrivial zero of $L(s,\pi)$, we defined

$$\mathcal{S}_{\pi,\Psi}(\rho_\pi) := \mathcal{S}_{\pi,\Psi}(X_1,X_2,T,\rho_\pi) = \sum_{|\operatorname{Im} \rho_{\pi,1}| \leqslant T} \sum_{|\operatorname{Im} \rho_{\pi,2}| \leqslant T} \hat{\Psi}_{X_1}(\rho_{\pi,1} - \rho_\pi)\hat{\Psi}_{X_2}(\rho_\pi - \rho_{\pi,2}), \tag{8.1}$$

where $\Psi_X(x)$ is defined by (1.27), and the sums run over the nontrivial zeros $\rho_{\pi,1}, \rho_{\pi,2}$ of $L(s,\pi)$. Throughout this section, when the parameters X_1, X_2, and T are clear from the context, we will omit them from the notation. We begin our analysis with the following lemma.

Lemma 8.1. *Let $m \in \mathbb{N}$ and $\pi \in \mathscr{A}_m$. Assume Hypothesis $\mathbf{H}_\pi$ and the Riemann Hypothesis for $L(s,\pi)$. Let $\Psi \in C_c^\infty(0,\infty)$ be a fixed non-negative compactly supported smooth function. Suppose $T \geqslant 2$, $\alpha_1, \alpha_2 > 0$,*

$X_1 = T^{\alpha_1 m}$ and $X_2 = T^{\alpha_2 m}$. Then for all non-trivial zeros $\rho_\pi = \beta_\pi + i\gamma_\pi$ of $L(s,\pi)$ with $|\gamma_\pi| \leqslant T$ outside an exceptional set of size at most $O_\pi(\sqrt{T}\log T)$, we have

$$\mathcal{S}_{\pi,\Psi}(\rho_\pi) = \sum_{n_1,n_2=1}^{\infty} \frac{\Psi_{X_1}(n_1)\Psi_{X_2}(n_2)\Lambda_\pi(n_1)\Lambda_{\widetilde{\pi}}(n_2)}{(n_1n_2)^{\frac{1}{2}}}\left(\frac{n_2}{n_1}\right)^{i\gamma_\pi} + O_{\pi,\Psi,\alpha_1,\alpha_2}\left(\frac{X_1^{1/2}\log T}{T^{1/2}} + \frac{X_2^{1/2}\log T}{T^{1/2}}\right).$$

Proof. By the functional equation (2.2) for $L(s,\pi)$, we rewrite (8.1) as

$$\mathcal{S}_{\pi,\Psi}(\rho_\pi) = \sum_{|\operatorname{Im}\rho_{\pi,1}|\leqslant T} \hat{\Psi}_{X_1}(\rho_{\pi,1} - \rho_\pi) \sum_{|\operatorname{Im}\rho_{\widetilde{\pi},2}|\leqslant T} \hat{\Psi}_{X_2}(\rho_{\widetilde{\pi},2} - \rho_{\widetilde{\pi}}). \tag{8.2}$$

By (7.4), for all non-trivial zeros $\rho_\pi = \beta_\pi + i\gamma_\pi'$ of $L(s,\pi)$ with $|\gamma_\pi| \leqslant T$ outside an exceptional set of size at most $O_\pi(\sqrt{T}\log T)$, we have

$$\sum_{|\operatorname{Im}\rho_{\pi,1}|\leqslant T} \hat{\Psi}_{X_1}(\rho_{\pi,1} - \rho_\pi) = -\sum_{n_1=1}^{\infty} \frac{\Psi_{X_1}(n_1)\Lambda_\pi(n_1)}{n_1^{\rho_\pi}} + O_{\pi,\Psi,\alpha_1}\left(\frac{\log T}{\sqrt{T}}\right) \tag{8.3}$$

$$\text{and} \quad \sum_{|\operatorname{Im}\rho_{\widetilde{\pi},2}|\leqslant T} \hat{\Psi}_{X_2}(\rho_{\widetilde{\pi},2} - \rho_{\widetilde{\pi}}) = -\sum_{n_2=1}^{\infty} \frac{\Psi_{X_2}(n_2)\Lambda_{\widetilde{\pi}}(n_2)}{n_2^{\rho_{\widetilde{\pi}}}} + O_{\pi,\Psi,\alpha_2}\left(\frac{\log T}{\sqrt{T}}\right). \tag{8.4}$$

Substituting (8.3) and (8.4) in (8.2) and applying Hypothesis $\mathbf{H}_\pi$, we arrive at

$$\mathcal{S}_{\pi,\Psi}(\rho_\pi) = \sum_{n_1,n_2=1}^{\infty} \frac{\Psi_{X_1}(n_1)\Psi_{X_2}(n_2)\Lambda_\pi(n_1)\Lambda_{\widetilde{\pi}}(n_2)}{(n_1n_2)^{\frac{1}{2}}}\left(\frac{n_2}{n_1}\right)^{i\gamma_\pi} + O_{\pi,\Psi,\alpha_1,\alpha_2}\left(\frac{X_1^{1/2}\log T}{T^{1/2}} + \frac{X_2^{1/2}\log T}{T^{1/2}}\right),$$

which completes the proof. □

8.2. **Moment Computations.** We now study moments of sums over primes. Fix $\xi > 1$. Let $T \geqslant 2$ and $\alpha_1, \alpha_2, X_1, X_2, V_1$ and V_2 satisfy

$$0 < \alpha_1, \alpha_2 < \frac{1}{m}, \quad X_1 = T^{\alpha_1 m}, \quad X_2 = T^{\alpha_2 m}, \quad V_1 = X_1^{\xi} \quad \text{and} \quad V_2 = X_2^{\xi}. \tag{8.5}$$

Let $\Psi \in C_c^\infty(0,\infty)$ be a fixed non-negative compactly supported smooth function. Define

$$\mathcal{G}_{\pi,\Psi}(\rho_\pi) = \mathcal{G}_{\pi,\Psi}(X_1,X_2,\rho_\pi) := \sum_{\substack{n_1\leqslant V_1\\ n_2\leqslant V_2}} \frac{\Psi_{X_1}(n_1)\Psi_{X_2}(n_2)\Lambda_\pi(n_1)\Lambda_{\widetilde{\pi}}(n_2)}{(n_1n_2)^{\frac{1}{2}}}\left(\frac{n_2}{n_1}\right)^{i\gamma_\pi}, \tag{8.6}$$

$$\mathcal{G}^*_{\pi,\Psi}(\rho_\pi) = \mathcal{G}^*_{\pi,\Psi}(X_1,X_2,\rho_\pi) := \sum_{\substack{p_1\leqslant V_1\\ p_2\leqslant V_2}} \frac{\Psi_{X_1}(p_1)\Psi_{X_2}(p_2)\Lambda_\pi(p_1)\Lambda_{\widetilde{\pi}}(p_2)}{(p_1p_2)^{\frac{1}{2}}}\left(\frac{p_2}{p_1}\right)^{i\gamma_\pi}, \tag{8.7}$$

where p_1, p_2 runs over primes up to V_1 and V_2 respectively.

Lemma 8.2. *Let $m \in \mathbb{N}, \pi \in \mathscr{A}_m$ and $\theta_m \in [0, \frac{1}{2} - \frac{1}{m^2+1}]$ be an admissible exponent towards the Ramanujan conjecture for $L(s,\pi)$. Assume Hypothesis $\mathbf{H}_\pi$ and the Riemann Hypothesis for $L(s,\pi)$. Let $\Psi \in C_c^\infty(0,\infty)$ be a fixed non-negative compactly supported smooth function. Suppose k, ℓ be non-negative integers with $k + \ell = 2d$ for some $d \in \mathbb{N}$. Let $T \geqslant 2$ and $\alpha_1, \alpha_2, \xi, X_1, X_2, V_1, V_2$ be as defined in* (8.5) *satisfying $\alpha_1 \neq \alpha_2$ and*

$$\alpha_1 + \alpha_2 < \frac{1}{2dm(1+\xi\theta_m)}.$$

Then

$$\frac{1}{N_\pi(T)}\sum_{|\gamma_\pi|\leqslant T}\mathcal{G}^*_{\pi,\Psi}(\rho_\pi)^k\overline{\mathcal{G}^*_{\pi,\Psi}(\rho_\pi)}^\ell = \boldsymbol{\delta}_{k\ell}d!^2\bigg(\sum_{p\leqslant V_1}\frac{\Psi^2_{X_1}(p)|\Lambda_\pi(p)|^2}{p}\bigg)^d\bigg(\sum_{p\leqslant V_2}\frac{\Psi^2_{X_2}(p)|\Lambda_\pi(p)|^2}{p}\bigg)^d \tag{8.8}$$

$$+ O_{\pi,\Psi,d,\alpha_1,\alpha_2,\xi}(\max\{1,(\log T)^{2d-2}\}). \tag{8.9}$$

Here $\mathcal{G}^*_{\pi,\Psi}(\rho_\pi)$ *is defined by* (8.7) *and* $\boldsymbol{\delta}_{k\ell}=1$ *if* $k=\ell$*, and zero otherwise.*

Proof. The proof is similar to the ideas in Lemma 3.1. All implied constants in the proof may depend at most on $\pi, d, \alpha_1, \alpha_2$ and ξ. We assume T is sufficiently large depending on $\pi, d, \alpha_1, \alpha_2$ and ξ. For $k, \ell \in \mathbb{N}$, let $\vec{p_i} = (p_{i1}, p_{i2})$ for $1 \leqslant i \leqslant k$ and $\vec{q_j} = (q_{j1}, q_{j2})$ for $1 \leqslant j \leqslant \ell$. We let $P_1 = \prod_{i=1}^k p_{i1}, P_2 = \prod_{i=1}^k p_{i2}, Q_1 = \prod_{j=1}^\ell q_{j1}$ and $Q_1 = \prod_{j=1}^\ell q_{j2}$. We write the left hand side of (8.8) as

$$\sum_{\substack{\vec{p_1},\dots,\vec{p_k}\\ \vec{q_1},\dots,\vec{q_\ell}}}\frac{\prod_{i=1}^k\Psi_{X_1}(p_{i1})\Psi_{X_2}(p_{i2})\Lambda_\pi(p_{i1})\Lambda_{\tilde{\pi}}(p_{i2})\prod_{j=1}^\ell\Psi_{X_1}(q_{j1})\Psi_{X_2}(q_{j2})\Lambda_{\tilde{\pi}}(q_{j1})\Lambda_\pi(q_{j2})}{(P_1P_2Q_1Q_2)^{\frac12}}$$

$$\times\frac{1}{N_\pi(T)}\sum_{|\gamma_\pi|\leqslant T}\bigg(\frac{P_2Q_1}{P_1Q_2}\bigg)^{i\gamma_\pi}, \tag{8.10}$$

where the outer sum in the right hand side of (8.10) is restricted to vectors $\vec{p_i}, \vec{q_j}$ with $2 \leqslant p_{i1}, q_{j1} \leqslant V_1$ and $2 \leqslant p_{i2}, q_{j2} \leqslant V_2$. We break our proof into two cases.

Case 1 : When $k \neq \ell$. In this situation, $(P_2Q_1)/(P_1Q_2)$ is never an integer. Similar to the proof of Case 1 in Lemma 3.1, we apply Corollaries 2.6 and 2.7 to the inner sum over γ_π in (8.10). Finally, by Cauchy–Schwarz and Lemma 2.11, the contribution to (8.10) from this case is

$$\ll_{\pi,\Psi,d,\alpha_1,\alpha_2,\xi,\varepsilon}\frac{(X_1X_2)^{2d(1+\xi\theta_m)+\varepsilon}\log T}{T}. \tag{8.11}$$

Case 2 : When $k = \ell$. The main terms occur when $P_2Q_1 = P_1Q_2$. For T large, this is only possible when $P_1 = Q_1$ and $P_2 = Q_2$. Applying a combinatorial argument similar to Lemma 3.1, the contribution to (8.10) when $P_2Q_1 = P_1Q_2$ is

$$d!^2\bigg(\sum_{p\leqslant V_1}\frac{\Psi^2_{X_1}(p)|\Lambda_\pi(p)|^2}{p}\bigg)^d\bigg(\sum_{p\leqslant V_2}\frac{\Psi^2_{X_2}(p)|\Lambda_\pi(p)|^2}{p}\bigg)^d + O_{\pi,\Psi,d,\alpha_1,\alpha_2,\xi}(\max\{1,(\log T)^{2d-2}\}). \tag{8.12}$$

The situation when $P_2Q_1 \neq P_1Q_2$ can be handled exactly as in Case 1.

Combining (8.11) and (8.12) and choosing $\varepsilon > 0$ sufficiently small, we arrive at our desired result. □

Lemma 8.3. *Let* $m \in \mathbb{N}, \pi \in \mathscr{A}_m$ *and* $\theta_m \in [0, \frac12 - \frac{1}{m^2+1}]$ *be an admissible exponent towards the Ramanujan conjecture for* $L(s,\pi)$*. Assume Hypothesis* $\mathbf{H}_\pi$ *and the Riemann Hypothesis for* $L(s,\pi)$*. Let* $\Psi \in C_c^\infty(0,\infty)$ *be a fixed non-negative compactly supported smooth function. Let* $d \in \mathbb{N}$*,* $T \geqslant 2$ *and* $\alpha_1, \alpha_2, \xi, X_1, X_2, V_1, V_2$ *be as in* (8.5) *satisfying* $\alpha_1 \neq \alpha_2$ *and*

$$\alpha_1+\alpha_2 < \frac{1}{2dm(1+\xi\theta_m)}.$$

Then

$$\frac{1}{N_\pi(T)}\sum_{|\gamma_\pi|\leqslant T}(\operatorname{Re}\mathcal{G}^*_{\pi,\Psi}(\rho_\pi))^{2d} = \boldsymbol{\mathcal{L}}_{2d}\bigg(\frac12\sum_{p\leqslant V_1}\frac{\Psi^2_{X_1}(p)|\Lambda_\pi(p)|^2}{p}\bigg)\bigg(\frac12\sum_{p\leqslant V_2}\frac{\Psi^2_{X_2}(p)|\Lambda_\pi(p)|^2}{p}\bigg)^d$$
$$+ O_{\pi,\Psi,d,\alpha_1,\alpha_2,\xi}(\max\{1,(\log T)^{2d-2}\}).$$

where $\boldsymbol{\mathcal{L}}_{2d}$ *is defined by* (1.35)*. Also, if*

$$\alpha_1+\alpha_2<\frac{1}{m(2d-1)(1+\xi\theta_m)},$$

then

$$\frac{1}{N_\pi(T)}\sum_{|\gamma_\pi|\leqslant T}(\operatorname{Re}\mathcal{G}^*_{\pi,\Psi}(\rho_\pi))^{2d-1}\ll_{\pi,\Psi,d,\alpha_1,\alpha_2,\xi}T^{-\varepsilon}\quad\textit{for some}\quad\varepsilon=\varepsilon(\pi,d,\alpha_1,\alpha_2,\xi)>0.$$

Proof. The proof precisely follows the arguments in Lemma 3.2. □

Let $m\in\mathbb{N}$ and $\pi\in\mathscr{A}_m$. Let $\Psi\in C_c^\infty(0,\infty)$ be a fixed non-negative compactly supported smooth function. Let $X_1,X_2,T\geqslant 2$ and $V_1=X_1^\xi, V_2=X_2^\xi$ for some fixed $\xi>1$. Define the following:

$$\widehat{\operatorname{Av}}_{\pi,\Psi}(T,X_1,X_2,\xi):=\frac{1}{N_\pi(T)}\sum_{|\gamma_\pi|\leqslant T}\operatorname{Re}\mathcal{G}_{\pi,\Psi}(\rho_\pi),\tag{8.13}$$

$$\widehat{\operatorname{Var}}_{\pi,\Psi}(T,X_1,X_2,\xi):=\Bigg(\frac12\sum_{n\leqslant V_1}\frac{\Psi^2_{X_1}(n)|\Lambda_\pi(n)|^2}{n}\Bigg)\Bigg(\frac12\sum_{n\leqslant V_2}\frac{\Psi^2_{X_2}(n)|\Lambda_\pi(n)|^2}{n}\Bigg)-\widehat{\operatorname{Av}}^2_{\pi,\Psi}.\tag{8.14}$$

We establish below asymptotic formulas for $\widehat{\operatorname{Av}}_{\pi,\Psi}$ and $\widehat{\operatorname{Var}}_{\pi,\Psi}$ without assuming RH for $L(s,\pi)$.

Lemma 8.4. *Let* $m\in\mathbb{N},\pi\in\mathscr{A}_m$ *and* $\theta_m\in[0,\frac12-\frac{1}{m^2+1}]$ *be an admissible exponent towards the Ramanujan conjecture for* $L(s,\pi)$*. Assume Hypothesis* $\mathbf{H}_\pi$ *for* $L(s,\pi)$*. Let* $\Psi\in C_c^\infty(0,\infty)$ *be a fixed non-negative compactly supported smooth function. Let* $d\in\mathbb{N}$*,* $T\geqslant 2$ *and* $\alpha_1,\alpha_2,\xi,X_1,X_2,V_1,V_2$ *be as in* (8.5) *satisfying* $\alpha_1\neq\alpha_2$ *and*

$$\alpha_1+\alpha_2<\frac{1}{m(1+\xi\theta_m)}.$$

Then

$$\widehat{\operatorname{Av}}_{\pi,\Psi}(T,X_1,X_2,\xi)\ll_{\pi,\Psi,\alpha_1,\alpha_2,\xi}(\log T)^{-1},\tag{8.15}$$

$$\textit{and}\quad\widehat{\operatorname{Var}}_{\pi,\Psi}(T,X_1,X_2,\xi)=\frac{\alpha_1\alpha_2(m\log T)^2}{4}\left(\int_0^\infty\frac{\Psi^2(t)}{t}\,\mathrm{d}t\right)^2+O_{\pi,\Psi,\alpha_1,\alpha_2,\xi}(\log T).\tag{8.16}$$

Proof. All implied constants in this proof depend only on π, α_1, α_2, and ξ. We assume T is sufficiently large, depending on π, α_1, α_2, and ξ. Without loss of generality, we suppose that $\alpha_2>\alpha_1$. By applying Corollary 2.7, we obtain the following expression for $\widehat{\operatorname{Av}}_{\pi,\Psi}$:

$$\widehat{\operatorname{Av}}_{\pi,\Psi}=-\frac{T}{\pi N_\pi(T)}\operatorname{Re}\sum_{\substack{n_1\leqslant V_1\\ n_2\leqslant V_2}}\frac{\Psi_{X_1}(n_1)\Psi_{X_2}(n_2)\Lambda_\pi(n_1)\Lambda_{\widetilde{\pi}}(n_2)}{n_2}\Lambda_\pi\left(\frac{n_2}{n_1}\right)+O_{\pi,\Psi,\xi}\left(\frac{(X_1X_2)^{1+\xi\theta_m+\varepsilon}\log T}{T}\right),\tag{8.17}$$

for any $\varepsilon>0$. The error term is acceptable by choosing ε sufficiently small. We note that $\Lambda_\pi(n_2/n_1)\neq 0$ if and only if n_2/n_1 is a prime power, that is, if for some prime p, $n_1=p^{\ell_1}$ and $n_2=p^{\ell_2}$ where $\ell_1,\ell_2\in\mathbb{N}$ with $\ell_2>\ell_1$. So we rewrite the double sum over n_1,n_2 as

$$\sum_{\substack{\ell_1,\ell_2=0\\ \ell_2>\ell_1}}^\infty\sum_{p\leqslant\min\{V_1^{1/\ell_1},V_2^{1/\ell_2}\}}\frac{\Psi_{X_1}(p^{\ell_1})\Psi_{X_2}(p^{\ell_2})\Lambda_\pi(p^{\ell_1})\Lambda_{\widetilde{\pi}}(p^{\ell_2})}{p^{\ell_2}}\Lambda_\pi(p^{\ell_2-\ell_1})\tag{8.18}$$

Note that the length of the sums over ℓ_1 and ℓ_2 in (8.18) is $\ll_{\pi,\Psi,\alpha_1,\alpha_2,\xi}1$. We consider two cases:

Case 1 : $\ell_2>m^2+1$**.** In this case, using (2.4), the sum over p and ℓ_2 converges absolutely. Therefore, (8.18) is $\ll_{\pi,\Psi,\alpha_1,\alpha_2,\xi}1$.

Case 2 : $\ell_2 \leqslant m^2+1$. Here we treat each tuple (ℓ_1, ℓ_2) separately. Fixing (ℓ_1, ℓ_2) and applying Cauchy–Schwarz, (8.18) is

$$\ll_{\pi,\Psi,\alpha_1,\alpha_2,\xi} \left(\sum_p \frac{|\Lambda_{\widetilde{\pi}}(p^{\ell_2})|^2}{p^{\ell_2}}\right)^{\frac{1}{2}} \left(\sum_p \frac{|\Lambda_\pi(p^{\ell_1})\Lambda_\pi(p^{\ell_2-\ell_1})|^2}{p^{\ell_2}}\right)^{\frac{1}{2}}. \tag{8.19}$$

Using Hypothesis $\mathbf{H}_\pi$, it follows that (8.19) is $\ll_{\pi,\Psi,\alpha_1,\alpha_2,\xi} 1$.

Combining Cases 1 and 2 in (8.17) and applying (2.8), we deduce (8.15) Applying Lemma 7.1 and using (8.15), we then conclude the proof of (8.16). ☐

We are now ready to estimate our moments involving sums over prime powers.

Lemma 8.5. *Let $m \in \mathbb{N}, \pi \in \mathscr{A}_m$ and $\theta_m \in [0, \frac{1}{2} - \frac{1}{m^2+1}]$ be an admissible exponent towards the Ramanujan conjecture for $L(s,\pi)$. Assume Hypothesis $\mathbf{H}_\pi$ and the Riemann Hypothesis for $L(s,\pi)$. Let $\Psi \in C_c^\infty(0,\infty)$ be a fixed non-negative compactly supported smooth function. Let $d \in \mathbb{N}$, $T \geqslant 2$ and $\alpha_1, \alpha_2, \xi, X_1, X_2, V_1, V_2$ be as in* (8.5) *satisfying $\alpha_1 \neq \alpha_2$ and*

$$\alpha_1 + \alpha_2 < \frac{1}{2dm(1+\xi\theta_m)}.$$

Then

$$\frac{1}{N_\pi(T)} \sum_{|\gamma_\pi| \leqslant T} (\operatorname{Re} \mathcal{G}_{\pi,\Psi}(\rho_\pi))^{2d} = \boldsymbol{\mathcal{L}}_{2d} \widehat{\operatorname{Var}}^{d}_{\pi,\Psi} + O_{\pi,\Psi,d,\alpha_1,\alpha_2,\xi}((\log T)^{2d-\frac{1}{2}}\}).$$

where $\boldsymbol{\mathcal{L}}_{2d}$ is given by (1.35) *and $\widehat{\operatorname{Var}}_{\pi,\Psi}$ is defined by* (8.14). *Also, if*

$$\alpha_1 + \alpha_2 < \frac{1}{m(2d-1)(1+\xi\theta_m)},$$

then

$$\frac{1}{N_\pi(T)} \sum_{|\gamma_\pi| \leqslant T} (\operatorname{Re} \mathcal{G}_{\pi,\Psi}(\rho_\pi))^{2d-1} \ll_{\pi,\Psi,d,\alpha_1,\alpha_2,\xi} (\log T)^{d-1}.$$

Proof. All implied constants in the proof may depend at most on $\pi, d, \alpha_1, \alpha_2$ and ξ. Our proof strategy is similar to that of Lemma 3.4. Therefore, we only highlight the necessary differences. First consider the case when the exponent is even. We break the double sum in $\mathcal{G}_{\pi,\Psi}$ as follows:

$$\begin{aligned} \mathcal{G}_{\pi,\Psi}(\rho_\pi) &= \sum_{p_1 \leqslant V_1} \sum_{p_2 \leqslant V_2} + \sum_{p_1 \leqslant V_1} \sum_{\substack{n_2 = p_2^k \leqslant V_2 \\ k \geqslant 2}} + \sum_{\substack{n_1 = p_1^k \leqslant V_1 \\ k \geqslant 2}} \sum_{p_2 \leqslant V_1} + \sum_{\substack{n_1 = p_1^k \leqslant V_1 \\ k \geqslant 2}} \sum_{\substack{n_2 = p_2^\ell \leqslant V_2 \\ \ell \geqslant 2}} \\ &= \mathcal{G}^*(\rho_\pi) + \mathcal{G}_1(\rho_\pi) + \mathcal{G}_2(\rho_\pi) + \mathcal{G}_3(\rho_\pi), \end{aligned} \tag{8.20}$$

where $\mathcal{G}^*$ is defined by (8.7), p_1, p_2 run over primes and we drop the suffix π, Ψ for brevity. We write

$$\begin{aligned} \frac{1}{N_\pi(T)} \sum_{|\gamma_\pi| \leqslant T} (\operatorname{Re} \mathcal{G}_{\pi,\Psi}(\gamma_\pi))^{2d} &= \frac{1}{N_\pi(T)} \sum_{\substack{j_1,j_2,j_3,j_4=0 \\ j_1+j_2+j_3+j_4=2d}}^{2d} \binom{2d}{j_1}\binom{2d}{j_2}\binom{2d}{j_3}\binom{2d}{j_4} \\ &\times \sum_{|\gamma_\pi| \leqslant T} (\operatorname{Re} \mathcal{G}^*(\rho_\pi))^{j_1} (\operatorname{Re} \mathcal{G}_1(\rho_\pi))^{j_2} (\operatorname{Re} \mathcal{G}_2(\rho_\pi))^{j_3} (\operatorname{Re} \mathcal{G}_3(\rho_\pi))^{j_4}. \end{aligned} \tag{8.21}$$

By Lemma 8.3 and Hypothesis $\mathbf{H}_\pi$ (see [47, Eq. 2.25]), the contribution to (8.21) from $j_1 = 2d$ is

$$\boldsymbol{\mathcal{L}}_{2d} \widehat{\operatorname{Var}}^{d}_{\pi,\Psi} + O_{\pi,\Psi,d,\alpha_1,\alpha_2,\xi}((\log T)^{2d-1}). \tag{8.22}$$

The cases $j_2 = 2d$ and $j_3 = 2d$ are symmetric. So we only consider $j_2 = 2d$. We break into cases when $k > m^2 + 1$ and otherwise. We write

$$\Big|\sum_{p_1\leqslant V_1}\sum_{\substack{p_2\leqslant V_2\\ n_2=p_2^k,k\geqslant 2}}\Big|^{2d} \ll_d \Big|\sum_{p_1\leqslant V_1}\sum_{\substack{n_2=p_2^k\leqslant V_2\\ 2\leqslant k\leqslant m^2+1}}\Big|^{2d} + \Big|\sum_{p_1\leqslant V_1}\sum_{\substack{n_2=p_2^k\leqslant V_2\\ k>m^2+1}}\Big|^{2d}. \tag{8.23}$$

Since

$$\sum_{k>m^2+1}\sum_{\substack{n_2\leqslant V_2\\ n_2=p_2^k}} \frac{\Psi_{X_2}(n_2)\Lambda_\pi(n_2)}{n^{\frac{1}{2}-i\gamma_\pi}} \ll_{\pi,\Psi,\xi} 1,$$

the second term in the right hand side of (8.23) can be addressed as follows:

$$\frac{1}{N_\pi(T)}\sum_{|\gamma_\pi|\leqslant T}\Big|\sum_{p_1\leqslant V_1}\sum_{\substack{n_2=p_2^k\leqslant V_2\\ k>m^2+1}}\Big|^{2d} \ll_{\pi,\Psi,d,\alpha_1,\alpha_2,\xi} \frac{1}{N_\pi(T)}\sum_{|\gamma_\pi|\leqslant T}\Big|\sum_{p_1\leqslant V_1}\Big|^{2d} \ll_{\pi,\Psi,d,\alpha_1,\alpha_2,\xi} (\log T)^d, \tag{8.24}$$

where we apply Lemma 7.6 to obtain the final inequality. Now we treat the first term in the right hand side of (8.23). To this end, we first apply Jensen's inequality to obtain

$$\frac{1}{N_\pi(T)}\sum_{|\gamma_\pi|\leqslant T}\Big|\sum_{p_1\leqslant V_1}\sum_{\substack{n_2=p_2^k\leqslant V_2\\ 2\leqslant k\leqslant m^2+1}}\Big|^{2d} \ll_{\pi,\Psi,d,\alpha_1,\alpha_2,\xi} \sum_{k=2}^{m^2+1}\frac{1}{N_\pi(T)}\sum_{|\gamma_\pi|\leqslant T}\Big|\sum_{p_1\leqslant V_1}\sum_{n_2=p_2^k\leqslant V_2}\Big|^{2d}, \tag{8.25}$$

For each fixed k, we first expand the $2d$-th power in the right hand side of (8.25). Suppose there exists some $2 \leqslant k_0 \leqslant m^2 + 1$ such that $\alpha_1 = \alpha_2/k_0$. When $k \neq k_0$, we follow the argument from Lemma 8.2. The combinatorial argument in this scenario is precisely the same and thus, the contribution to the right side of (8.25) when $k \neq k_0$ is

$$\ll_{\pi,\Psi,d,\alpha_1,\alpha_2,\xi} \sum_{\substack{k=2\\ k\neq k_0}}^{m^2+1}\Big(\sum_{p\leqslant V_1}\frac{\Psi_{X_1}^2(p)|\Lambda_\pi(p)|^2}{p}\Big)^d\Big(\sum_{p^k\leqslant V_2}\frac{\Psi_{X_2}^2(p^k)|\Lambda_\pi(p^k)|^2}{p^k}\Big)^d \ll_{\pi,\Psi,d,\alpha_1,\alpha_2,\xi} (\log T)^d, \tag{8.26}$$

by Hypothesis $\mathbf{H}_\pi$. The case $k = k_0$ is only considered if such k_0 exists. The argument in this case follows as in Case 2 of Lemma 8.4. In particular, the contribution to (8.25) from this case is

$$\ll_{\pi,\Psi,d,\alpha_1,\alpha_2,\xi}\Big(\sum_{p\leqslant V_1}\frac{\Psi_{X_1}^2(p)|\Lambda_\pi(p)|^2}{p}\Big)^d\prod_{j=1}^d \mathcal{T}_j, \tag{8.27}$$

where $\displaystyle \mathcal{T}_j = \sum_{p^{k_0}\leqslant V_2}\frac{\Psi_{X_2}^2(p^{k_0})|\Lambda_\pi(p^{k_0})|^2}{p^{k_0}}$ or $\displaystyle \sum_{p^{k_0}\leqslant V_2}\frac{\Psi_{X_1}^{k_0}(p)\Psi_{X_2}(p^{k_0})|\Lambda_{\widetilde{\pi}}(p)|^{k_0}|\Lambda_\pi(p^{k_0})|}{p^{k_0}}, 1\leqslant j\leqslant d.$

By Cauchy–Schwarz, Hypothesis $\mathbf{H}_\pi$ and Lemma 2.11, it follows that each $\mathcal{T}_j \ll_{\pi,\Psi,d,\alpha_1,\alpha_2,\xi} 1$. This implies that (8.27) is $\ll_{\pi,\Psi,d,\alpha_1,\alpha_2,\xi} (\log T)^d$. Putting this together with (8.24) and (8.26), the overall contribution when $j_2 = 2d$ is $\ll_{\pi,\Psi,d,\alpha_1,\alpha_2,\xi} (\log T)^d$. The same bound holds when $j_3 = 2d$. Finally, techniques similar to the case $j = 0$ in Lemma 3.4 and the cases $j_2 = 2d, j_3 = 2d$ show that the contribution from $j_4 = 2d$ is $O_{\pi,\Psi,d,\alpha_1,\alpha_2,\xi}(1)$. We now apply a generalized version of Hölder's inequality to deal with the other tuples (j_1, j_2, j_3, j_4) in (8.21) to arrive at our desired result. In the case of odd moments, the argument is precisely as in Lemma 3.4 with necessary applications of Lemma 8.3. ☐

Remark 8.6. Suppose $f : \mathbb{R} \to \mathbb{R}$ be any function satisfying $f(T) \to \infty$ as $T \to \infty$. Assume that $\alpha_1, \alpha_2 > 0$ such that for $T \geqslant 2$ sufficiently large, we have

$$|\alpha_1 - \alpha_2| \geqslant \frac{f(T)}{\log T}. \tag{8.28}$$

Observe that the combinatorial analysis corresponding to Case 2 in the proof of Lemma 8.2 remains the same if we replace the condition $\alpha_1 \neq \alpha_2$ with the condition (8.29). We may now adapt the arguments in the subsequent lemmas accordingly.

8.3. **The case when $\alpha_1 = \alpha_2$.** In the special case where $\alpha_1 = \alpha_2 = \alpha$, our techniques remain the same as in Lemma 8.2 and the following computations. But the combinatorial analysis is different. Considering Lemma 8.2, for any non-negative integers k, ℓ with $k + \ell = r$ for some $r \in \mathbb{N}$, if all the primes in P_1, P_2, Q_1, and Q_2 are distinct, there are $r!$ possible ways to permute the primes among themselves. In particular, we remark here that the odd moments also contribute main terms. Hence in this scenario, writing $X = T^{\alpha}$ and $V = X^{\xi}$, we obtain

$$\frac{1}{N_\pi(T)} \sum_{|\gamma_\pi| \leqslant T} \mathcal{G}^*_{\pi,\Psi}(\rho_\pi)^k \overline{\mathcal{G}^*_{\pi,\Psi}(\rho_\pi)}^{\ell} = r! \bigg(\sum_{p \leqslant V} \frac{\Psi_X^2(p) |\Lambda_\pi(p)|^2}{p} \bigg)^r + O_{\pi,\Psi,d,\alpha,\xi}(\max\{1, (\log T)^{r-2}\}).$$

An application of Lemma 2.8, then implies that

$$\frac{1}{N_\pi(T)} \sum_{|\gamma_\pi| \leqslant T} (\operatorname{Re} \mathcal{G}^*_{\pi,\Psi}(\rho_\pi))^r = \boldsymbol{\chi}_r \bigg(\frac{1}{2} \sum_{p \leqslant V} \frac{\Psi_X^2(p) |\Lambda_\pi(p)|^2}{p} \bigg)^r + O_{\pi,\Psi,d,\alpha,\xi}(\max\{1, (\log T)^{r-2}\}).$$

From here on, the arguments in Lemma 8.5 can be adapted with necessary changes. Additionally, suppose $f : \mathbb{R} \to \mathbb{R}$ be any function satisfying $f(T) \to \infty$ as $T \to \infty$. Then the analysis remains the same if we replace the condition $\alpha_1 = \alpha_2$ by

$$|\alpha_1 - \alpha_2| \leqslant \frac{1}{f(T) \log T} \tag{8.29}$$

for $T \geqslant 2$ sufficiently large.

8.4. **Proof of Theorem 1.16.** Assume $\alpha_1 \neq \alpha_2$. First consider the case when the exponent is even and write $r = 2d$. We let $X_1 = T^{\alpha_1 m}, X_2 = T^{\alpha_2 m}$ and choose $V_1 = X_1^{\xi}, V_2 = X_2^{\xi}$ for some $\xi = \xi(\pi, d, \alpha_1, \alpha_2) > 1$ such that

$$\alpha_1 + \alpha_2 < \frac{1}{2dm(1 + \xi\theta_m)}.$$

Then Lemma 8.1 shows that for all non-trivial zeros $\rho_\pi = \frac{1}{2} + i\gamma_\pi$ of $L(s, \pi)$ with $|\gamma_\pi| \leqslant T$ outside an exceptional set of size at most $O_\pi(\sqrt{T} \log T)$, we have

$$\operatorname{Re} \mathcal{S}_{\pi,\Psi}(\rho_\pi) = \operatorname{Re} \mathcal{G}_{\pi,\Psi}(\rho_\pi) + \widehat{E}_{\pi,\Psi}(\rho_\pi), \tag{8.30}$$

where $\widehat{E}_{\pi,\Psi}(\rho_\pi) \ll_{\pi,\Psi,d,\alpha_1,\alpha_2} T^{-\delta_1}$ for some $\delta_1 = \delta_1(\pi, \Psi, d, \alpha_1, \alpha_2) > 0$. Therefore we have

$$\begin{aligned} \frac{1}{N_\pi(T)} \sum_{|\gamma_\pi| \leqslant T} (\operatorname{Re} \mathcal{S}_{\pi,\Psi}(\rho_\pi))^{2d} &= \frac{1}{N_\pi(T)} \sum_{|\gamma_\pi| \leqslant T} (\operatorname{Re} \mathcal{G}_{\pi,\Psi}(\rho_\pi) + \widehat{E}_{\pi,\Psi}(\rho_\pi))^{2d} \\ &\quad + \frac{1}{N_\pi(T)} \sum_{\rho_\pi \in \mathcal{E}} ((\operatorname{Re} \mathcal{S}_{\pi,\Psi}(\rho_\pi))^{2d} - (\operatorname{Re} \mathcal{G}_{\pi,\Psi}(\rho_\pi) + \widehat{E}_{\pi,\Psi}(\rho_\pi))^{2d}) \\ &= \widehat{\mathcal{U}}_1 + \widehat{\mathcal{U}}_2, \end{aligned}$$

say, where $\mathcal{E}$ is an exceptional set of non-trivial zeros $\rho_\pi = \beta_\pi + i\gamma_\pi$ of $L(s, \pi)$ with $|\gamma_\pi| \leqslant T$ and $\#\mathcal{E} \ll_\pi \sqrt{T} \log T$. An application of Lemma 8.5 shows that

$$\widehat{\mathcal{U}}_1 = \boldsymbol{\mathcal{L}}_{2d} \widehat{\operatorname{Var}}_{\pi,\Psi}^{d} + O_{\pi,\Psi,d,\alpha_1,\alpha_2}((\log T)^{2d - \frac{1}{2}})$$

where $\mathcal{L}_{2d}$ is given by (1.19) and $\widehat{\mathrm{Var}}_{\pi,\Psi}$ is as defined in (8.14). To bound $\widehat{\mathcal{U}}_2$, we use arguments similar to how we proved (4.2). For clarity, we mention that the term $\operatorname{Re}\mathcal{S}_{\pi,\Psi}(\rho_\pi)$ is bounded using rapid decay of the Mellin-transform and standard estimates on sums over zeros (see [22]). The term $\operatorname{Re}\mathcal{G}_{\pi,\Psi}(\rho_\pi)$ is bounded using Cauchy–Schwarz and Lemma 7.1. In conclusion, we have $\widehat{\mathcal{U}}_2 \ll_{\pi,\Psi,d,\alpha_1,\alpha_2} T^{-\delta_2}$ for some $\delta_2 = \delta_2(\pi,\Psi,d,\alpha_1,\alpha_2) > 0$. Combining the estimates for $\widehat{\mathcal{U}}_1$ and $\widehat{\mathcal{U}}_2$, we obtain

$$\frac{1}{N_\pi(T)} \sum_{|\gamma_\pi|\leqslant T} (\operatorname{Re}\mathcal{S}_{\pi,\Psi}(X_1,X_2,T,\rho_\pi))^{2d} = \mathcal{L}_{2d}\widehat{\mathrm{Var}}^{\,d}_{\pi,\Psi} + O_{\pi,\Psi,d,\alpha_1,\alpha_2}((\log T)^{2d-\frac{1}{2}}). \tag{8.31}$$

Applying (8.16), it follows that

$$\widehat{\mathcal{M}}_{\pi,\Psi,2d}(T^{\alpha_1 m}, T^{\alpha_2 m}, T) = \mathcal{L}_r\left(\frac{m\sqrt{\alpha_1\alpha_2}\log T}{2}\int_0^\infty \frac{\Psi^2(t)}{t}\,\mathrm{d}t\right)^{2d} + O_{\pi,\Psi,d,\alpha_1,\alpha_2}((\log T)^{2d-\frac{1}{2}}).$$

For odd moments, the proof is similar except that $\widehat{\mathcal{U}}_1 \ll_{\pi,d,\alpha_1,\alpha_2} (\log T)^{d-1}$.

When $\alpha_1 = \alpha_2$, the arguments are similar except that we use the necessary asymptotic estimates from Section 8.3 □

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