Content: Lectures 1-15

Lectures 1-7

1. Ordered Fields
    - Examples/Non-examples
    - Definition and properties of $\leq$
2. The Real Line
    - Definition and properties of the absolute value
    - Definition and properties of sup/inf
    - The completeness axiom
    - Archimedean property
    - Dense sets
3. Metric Spaces
    - Examples of metric spaces.
    - Norms induce metrics
    - Open/Closed sets in metric spaces
    - Bounded sets.
    - Interior, Closure, and Boundaries
    - Equivalent norms and metrics
4. Sequences in Metric Spaces
    - Bounded and convergent subsequences
    - Subsequences
5. Sequences in $\mathbb{R}$
    - Properties of convergent subsequences
    - Monotone sequences
    - Definition and properties of limsup/liminf
6. Completed Metric Spaces
    - Cauchy sequences
    - Definition of a complete metric space
    - $\mathbb{R}$ is complete
    - $\mathbb{R}^d$ is complete
7. Topological Spaces
    - Definition of a Topological Space $(X, \mathcal{T})$.
    - Metric Topology
    - Convergence in Topological spaces
    - Definition compact sets
    - Closed $\cap$ compact is closed
    - Nested sequences of compact sets have non-empty compact intersection.

8. Compact Sets in Metric Spaces.
    - Definition of Sequentially compact
    - Definition of Totally Bounded
    - examples and non-examples of seq
    - Compact $\implies$ Seq. Compact $\implies$ Totally Bounded and Complete $\implies$ Compact.
    - Heine-Borel Theorem.
9. Continuous functions
    - Definition and equivalent definitions
    - Cluster points
    - Images of compact sets are compact
    - Properties of real-values continuous functions (e.g. $f + g$ is continuous.)
10. Uniform continuity.
    - Definition
    - Examples/Non-examples
    - For uniformly continuous f, x_n is Cauchy $\implies$ $f(x_n)$ is Cauchy.
    - Continuous functions with compact domain are uniformly continuous.
11. Sequences of functions and types of convergence
    - Definition of Point-wise convergence
    - Example where the limit of sequence of functions is not continuous.
    - Definition of uniform convergence
    - Example of uniformly converging sequence of functions.
    - Uniform Cauchy Criterion
    - For complete co-domains, Uniformly Cauchy $\iff$ Uniform convergence
12. Connected sets
    - Definitions
    - $[0, 1]$ is connected.
    - Definition of path connected.
    - Images of connected sets are connected.
    - Examples of path connected and connected sets.
    - Path connected implies connected. Converse is not true.
    - Characterization of connected sets in $\mathbb{R}$.
    - Open and connected in $\mathbb{R}^n$ implies path connected.
    - The topologist's curve. (Do <u>NOT</u> need to know **how** to show it is connected but not path connected)

Practice Questions

1. For each question, say whether such an object exists. If it does, give an example. If it does not, give a short explanation on why not. Assume all the sequences are in $\mathbb{R}$.
   (a) A sequence which is not monotone and has no convergent subsequence.
   (b) A Cauchy sequence with a divergent subsequence.
   (c) A sequence with exactly two subsequential limits.
   (d) A bounded sequence with no Cauchy subsequence

2. True or False.
   (a) If (x_n) and (y_n) diverge, then the sequence $x_n y_n$ diverges.
   (b) If $(x_n) \to 0$ and (y_n) is bounded then $x_n y_n \to 0$.
   (c) Arbitrarily unions of closed sets are closed.
   (d) Closed and bounded sets are compact in metric spaces.
   (e) Let S be a nonempty bounded subset of $\mathbb{R}$ then $\inf S < \sup S$.

3. Prove the lower triangle inequality using the triangle inequality.

4. Use the fact that $\mathbb{Q}$ is dense to prove that there are infinitely many rationals in $[a, b]$ for $a, b \in R$.

5. Assuming that $\mathbb{Q}$ is dense show that for all $x \in \mathbb{R}$ there exists an n such that $n \leq x \leq n + 1$.

6. Let X be a set of real numbers that has a supremum, and let $a := \sup X$. Prove that $X \cap (a - \varepsilon, a] \neq \emptyset$ for every $\varepsilon > 0$.

7. Prove that
$$\limsup_{n\to\infty}(s_n + t_n) \leq \limsup_{n\to\infty} s_n + \limsup_{n\to\infty} y_n$$

8. Suppose that a_n and b_n are Cauchy sequences. Prove that $(a_n b_n)$ is a Cauchy sequence without using the fact that Cauchy sequences converge.

9. Let A and B be subsets of a metric space E. Prove that $\overline{A \cup B} = \overline{A} \cup \overline{B}$

10. Let $f : \mathbb{R} \to \mathbb{R}$ such that
$$|f(x) - f(y)| \leq |x - y|^2.$$
Prove that f is uniformly continuous.

11. Prove that for a bounded real sequence s_n that if $s_{n_k} \to L$ as $k \to \infty$ then
$$\liminf s_n \leq L \leq \limsup s_n.$$

12. Prove that for a real sequence s_n, $s_n \to \infty \iff s_n^{-1} \to 0$.

13. Let s_n be a sequence in a Metric space (E, d). Prove that if $\forall n$
$$d(s_n, s_{n+1}) < cd(s_n, s_{n-1})$$
for some $c \in (0, 1)$ then s_n is Cauchy.

14. Prove that $\lim n^{1/n} \to 1$.

15. Find the limit of
$$s_n = \frac{n^3 + 6n^2 + 7}{4n^3 + 3n - 4}$$
without using theorems from Calculus.

16. Prove that in a metric space all compact sets are totally bounded.
17. Let $f : X \to Y$ be a function such that for all compact $K \in Y$, $f^{-1}(K)$ is compact in X. Prove that f is continuous.
18. Which one of the following sets in compact in $\mathbb{R}$? Provide a brief explanation.
    (a) $(0, 1]$
    (b) $\{\frac{1}{n} : n \in \mathbb{N}\}$
    (c) $[1, \infty)$
19. True or False.
    (a) In a Metric space, Closed and bounded sets are compact.
    (b) $\mathbb{Q}$ is connected.
    (c) All compact sets are disconnected.
    (d) Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous and $A \subset \mathbb{R}$ is bounded then $f(A)$ is bounded.
    (e) If A and B are disconnected then $A \cap B$ is disconnected.
20. Let $\{f_n : (E, d) \to (E', d')\}$ be a sequence of uniformly continuous functions that converge uniformly to f. Prove that f is uniformly continuous.
21. Let $\{f_n : (E, d) \to (E', d')\}$ be a sequence of bounded functions that converge uniformly to f. Prove that f is bounded.
22. Prove that $x^3 = \cos(x)$ for some x in $(0, \pi/2)$.
23. Suppose that $g : \mathbb{R}^n \to \mathbb{R}^m$ is continuous at c and $g(c) \neq \hat{0}$ then there exists an open set U containing c such that $g(x) \neq 0$ for all $x \in U$.
24. Let X be a totally bounded space and $f : X - \mathbb{R}$ be uniformly continuous. Prove that $f(X)$ is totally bounded.
25. Give an example of sequence of functions that converge uniformly and prove that is uniformly convergent.