# Topics

This exam covers everything in the course. Here is a list of topics covered so far in the course. Last updated: **July 15, 2026**

1. **Ordered Fields**
    - Examples and non-examples
    - Definition and properties of $\leq$
2. **The Real Line**
    - Definition and properties of the absolute value
    - Definition and properties of supremum (sup) and infimum (inf)
    - The completeness axiom
    - Archimedean property
    - Dense sets
3. **Metric Spaces**
    - Examples of metric spaces
    - Norms induce metrics
    - Open and closed sets in metric spaces
    - Bounded sets
    - Interior, closure, and boundaries
    - Equivalent norms and metrics
4. **Sequences in Metric Spaces**
    - Bounded and convergent sequences
    - Subsequences
5. **Sequences in $\mathbb{R}$**
    - Properties of convergent sequences
    - Monotone sequences
    - Definition and properties of $\limsup$ and $\liminf$
6. **Complete Metric Spaces**
    - Cauchy sequences
    - Definition of a complete metric space
    - $\mathbb{R}$ is complete
    - $\mathbb{R}^d$ is complete
7. **Topological Spaces**
    - Definition of a topological space $(X, \mathcal{T})$
    - Metric topology
    - Convergence in topological spaces
    - Definition of compact sets
    - The intersection of a closed set and a compact set is compact

- Nested sequences of compact sets have a non-empty compact intersection

8. **Compact Sets in Metric Spaces**
   - Definition of sequentially compact
   - Definition of totally bounded
   - Examples and non-examples of sequential compactness
   - Compact $\implies$ Sequentially Compact $\implies$ Totally Bounded and Complete $\implies$ Compact
   - Heine-Borel Theorem
9. **Continuous Functions**
   - Definition and equivalent definitions
   - Cluster points
   - Images of compact sets are compact
   - Properties of real-valued continuous functions (e.g., $f + g$ is continuous)
10. **Uniform Continuity**
    - Definition
    - Examples and non-examples
    - For a uniformly continuous function $f$, $x_n$ is Cauchy $\implies$ $f(x_n)$ is Cauchy
    - Continuous functions with a compact domain are uniformly continuous
11. **Sequences of Functions and Types of Convergence**
    - Definition of pointwise convergence
    - Example where the limit of a sequence of functions is not continuous
    - Definition of uniform convergence
    - Example of a uniformly convergent sequence of functions
    - Uniform Cauchy Criterion
    - For complete codomains, Uniformly Cauchy $\iff$ Uniform convergence
12. **Connected Sets**
    - Definitions
    - $[0, 1]$ is connected
    - Definition of path-connected
    - Images of connected sets are connected
    - Examples of path-connected and connected sets
    - Path-connected implies connected; the converse is not true
    - Characterization of connected sets in $\mathbb{R}$
    - Open and connected sets in $\mathbb{R}^n$ imply path-connected
    - The topologist's sine curve (*Do <u>NOT</u> need to know **how** to show it is connected but not path-connected*)
13. **Differentiation**
    - Definition of the derivative
    - Differentiation rules

- Mean Value Theorem and Rolle's Theorem
- Corollaries of the MVT and Rolle's Theorem
- Inverse Function Theorem
- Taylor's Theorem and corollaries

14. **Integration**
    - Partitions
    - Upper and lower Darboux sums
    - Darboux integrability
    - Cauchy criterion for integrals
    - Riemann integrability
    - Continuous $\implies$ integrable
    - The integral is linear
    - The composition of a continuous function and an integrable function is integrable
    - Fundamental Theorems of Calculus
    - Logarithm and exponentiation
    - Change of variables
15. **Interchanging Limit Operations**
    - Uniform convergence and integrals
    - Uniform convergence and differentiation
    - Leibniz rule (*not examinable*)
16. **Series**
    - Definition
    - Absolute convergence and conditional convergence
    - Rearranging series
    - Tests of convergence (Comparison, Root, Ratio, and Integral)
17. **Power Series**
    - Definition
    - Examples from Taylor series
    - Radius of convergence
    - Weierstrass $M$-test
    - Integrals/Derivatives of power series
18. **Lebesgue Measure on $\mathbb{R}$**
    - Sums of sets
    - Properties of the Lebesgue measure
    - Definition of the Lebesgue outer measure
    - Properties of the Lebesgue outer measure
    - Definition of measurable sets
    - Intervals are measurable and their measure is their length
    - **Measurable Functions on $\mathbb{R}$**
      - Definition of measurable functions
      - Simple functions
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# Questions

1. Assume $s_n$ and $t_n$ are sequences of real numbers. If $s_n \to s > 0$, then

$$\limsup_{n\to\infty}(s_n t_n) = s \limsup_{n\to\infty} t_n.$$

   Why does it matter that $s \neq 0$?

2. Let $B$ be a topological space. Suppose $f : B \to \mathbb{R}$ is continuous. Prove or disprove: If $f(B)$ is connected, then $B$ is connected.

3. Let $(X, d)$ be a metric space and let $A \subseteq X$ be a connected subset. Prove that if $A \subseteq E \subseteq \bar{A}$, then $E$ is also connected.

4. Let $X, Y$, and $Z$ be metric spaces and $f : X \to Y$ and $g : Y \to Z$ be maps. Assume further that:
   - $X$ is compact;
   - $f$ is surjective and continuous; and
   - $g \circ f$ is continuous.

   Show that $g$ is continuous.

5. Determine whether the function

$$f(x) = |x|$$

   is differentiable at $x = 0$.

6. Suppose $f$ is differentiable on $(a, b)$ and continuous on $[a, b]$. Prove that if

$$f'(x) > 1 \quad \text{for all } x \in (a, b),$$

   then

$$f(b) - f(a) > b - a.$$

7. Let $f(x)$ be continuous at 0. Show that $xf(x)$ is differentiable at 0.

8. From the definition, prove that $x^2$ is Darboux integrable.

9. State and prove the Integral Test for series convergence.

10. Prove that the following series converge:
    (a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$
    (b) $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$
    (c) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$
    (d) $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$

11. Compute the radius of convergence of the following power series:
    (a) $\sum_{n=0}^{\infty} \frac{x^n}{n!}$
    (b) $\sum_{n=1}^{\infty} \frac{3^n}{\sqrt{n}} x^n$
    (c) $\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2} x^n$
    (d) $\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^{n^2} x^n$

12. Let $f$ be a bounded, integrable function from $[a, b] \to \mathbb{R}$. Prove that

$$F(x) := \int_a^x f(t)\,\mathrm{d}t$$

is continuous on $[a, b]$.

13. Consider the infinite series of functions defined on the interval $[0, \pi]$ by:

$$S(x) = \sum_{n=1}^{\infty} \frac{\sin(nx)}{n^2}$$

   (a) Prove that the series converges uniformly on $[0, \pi]$ to a continuous function $S(x)$. *(Hint: Use the Weierstrass M-test.)*

   (b) Justify why the integral and summation operations can be interchanged, and evaluate the integrated series explicitly:

$$\int_0^{\pi} \sum_{n=1}^{\infty} \frac{\sin(nx)}{n^2}\,\mathrm{d}x = \sum_{n=1}^{\infty} \int_0^{\pi} \frac{\sin(nx)}{n^2}\,\mathrm{d}x$$

14. Let $f : \mathbb{R} \to \mathbb{R}$ be a uniformly continuous function, and let

$$f_n(x) = \frac{n}{2} \int_{x-\frac{1}{n}}^{x+\frac{1}{n}} f(t)\,\mathrm{d}t.$$

Show that $f_n \to f$ uniformly on $\mathbb{R}$.

15. True/False:

   (a) $\mathbb{Q}$ is Lebesgue measurable and has positive measure.

   (b) All Lebesgue measure zero sets are countable.

   (c) Dense sets have non-zero Lebesgue measure.

   (d) All measurable functions are continuous.

16. Show that a continuous function between two metric spaces transforms path-connected sets into path-connected sets.

17. Compute from the definition of the derivative that the derivative of $f(x) = x^k$ is $kx^{k-1}$ for all $k \in \mathbb{N}$.

18. Prove that if $f : [a, b] \to \mathbb{R}$ is monotone, then $f$ is integrable.

19. Let $\{f_n\}$ and $\{g_n\}$ be sequences of real-valued functions on $D \subset \mathbb{R}$ that converge uniformly to $f$ and $g$ respectively.

   (a) Prove that $f_n + g_n$ converges uniformly to $f + g$.

   (b) Prove that if $f_n$ and $g_n$ are bounded, then $f_n g_n$ converges uniformly to $fg$.

   (c) Is the boundedness condition necessary?

20. If $(a_n)$ is a bounded sequence, then prove that the radius of convergence of the power series $\sum a_n x^n$ is at least 1.

21. Let $f(x)$ be an even function. Further, let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ have a positive radius of convergence. Prove that $a_k = 0$ for all odd $k$.

22. Finish the proof of the fact that rays are measurable.(The case when the cover is finite.)

23. Be able to state and prove all properties of the Lebesgue measure.

24. Let $\{E_n\}$ be am ascending nested sequence of Lebesgue measurable subsets prove that $m(E_n) \to m(\cup E_n)$.

25. Prove that countable interesections of measurable sets are measurable.

26. Let $\{f_n\}$ be a sequence of continuous functions from $\mathbb{R}$ to $\mathbb{R}$. Prove that the set $S = \{x \in \mathbb{R} \mid f_n(x) < 1 \text{ for all } n \in \mathbb{N}\}$ is measurable.