1. By means of a counterexample, show that $\mathbb{C}$ is not ordered.
2. Let $a, b, c \in \mathbb{R}$. If $b < c$ then prove that $a + b < b + c$.
3. Prove that $\sqrt{2} \in \mathbb{R}$ but that $\sqrt{2} \notin \mathbb{Q}$.
4. Prove that a norm induces a metric and give an example of metric that is not induced by a norm. What properties of a distance function are needed to be induced a norm?
5. Let $F = \mathbb{R}$ or $\mathbb{C}$ and V be a F-vector space. Prove that for $v_1, v_2, \ldots, v_n \in V$
$$|v_1 + v_2 + \cdots + v_n| \leq |v_1| + |v_2| + \cdots + |v_n|.$$
6. Let S and T be bounded non-empty subsets of $\mathbb{R}$. If $S \subseteq T$ prove that
$$\inf T \leq \inf S \leq \sup S \leq \sup T.$$
7. Prove that for any real $x < y \in \mathbb{R}$ there exists an irrational number r such that $x < r < y$.
8. For $x, y \in \mathbb{R}$, define $d(x, y) := \frac{|x-y|}{1+|x-y|}$. Does this give a metric on $\mathbb{R}$?
9. For each positive real number a show that for each natural number n there is a unique positive real x such that $x^n = a$.
10. For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ define
$$\|x\|_1 = |x_1| + |x_2| + \ldots + |x_n|$$
and
$$\|x\|_\infty = \sup\{|x_1|, \ldots, |x_n|\}.$$
Show that there are constants a and b such that
$$a\|x\|_1 \leq \|x\|_\infty \leq b\|x\|_1.$$
Find the largest a and the smallest b such that these inequalities always hold.

Discussion

Let $x := (x_n)_{n=1}^\infty$ be a sequence of complex numbers. For $p \geq 1$, we define the ℓ^p-norm as
$$||x||_p := \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p}.$$
When $||x||_p < \infty$, we say $x \in \ell^p$. We will show that ℓ^p is a normed space.

1. Show that $\|x\|_p > 0$ if x is not the sequence of all zeros.
2. Show that $\|\alpha x\|_p = |\alpha|\|x\|_p$ for all $\alpha \in \mathbb{C}$.
3. Prove Young's inequality: If $a, b \geq 0$ and if $p, q > 1$ with $1/p + 1/q = 1$, then
$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$
4. Use Young's inequality to prove Hölder's inequality: If $p, q \geq 1$ with $1/p + 1/q = 1$, then for $x \in \ell^p$ and $y \in \ell^q$
$$||xy||_1 \leq ||x||_p ||y||_p. \tag{1}$$

5. Use Hölder's inequality to prove Minkowski's Inequality (Triangle Inequality): For $x, y \in \ell^p$

$$||x+y||_p \leq ||x||_p + ||y||_p.$$

6. Give some examples of sequences that are in ℓ^2 but not in ℓ^1.

7. Show that for all $1 \leq q \leq p$, if $x \in \ell^q$ then $x \in \ell^p$.