1. What are some other formulations of denseness?
2. Let (a_n) and (b_n) be sequences in $\mathbb{R}$ with limits a and b, respectively. Show the following
   (a) $a_n + b_n \to a + b$
   (b) $a_n b_n \to ab$
   (c) $ca_n \to ca \quad \forall c \in \mathbb{R}$
   (d) If $b \neq 0$ and $\forall n$ large enough $b_n \neq 0$ then $\frac{a_n}{b_n} \to \frac{a}{b}$.
   (e) If $a_n \leq b_n \quad \forall n$ then $a \leq b$.
3. Let (E, d) be a metric space and $S \subseteq E$. Then prove that
   (a) $E - \bar{S} = (E - S)^\circ$
   (b) $\bar{S}$ is the set of all the limit points of S.
   (c) $\partial S = (E - S^\circ) \cap (E - (E - S)^\circ)$
4. Prove that for a bounded sequence of real numbers (a_n) the sequence $b_N := \sup\{a_n : n \geq N\}$ is monotone.
5. For $|r| < 1$, show that $s_n := \sum_{k=1}^{n} r^n$ converges.
6. (a) Let (s_n) be a sequence such that

$$|s_{n+1} - s_n| \leq 2^{-n} \quad \forall n \in \mathbb{N}. \tag{1}$$

   Prove that (s_n) is Cauchy.
   (b) Can we relax this condition i.e. change 2^{-n}?
7. Show that the sequence $a_n := \sum_{k=1}^{m} \frac{\sin(\pi k)}{2^k}$ is convergent.

Discussion

The metric topology on subsets

1. Let (E, d) be a metric space and $F \subset E$. Prove that (F, d_F) where d_F is the metric restricted to F is a metric space as well.
2. Let U be an open set in (E, d) prove that $F \cap S$ is open in (F, d_F).
3. Prove that for an open $U \subset F$ there exists an open subset V of E such that $U = V \cap F$.
4. Draw $S^n := \{x \in \mathbb{R}^{n+1} : ||x|| = 1\}$ for $n = 0, 1, 2$. Also, draw what an open ball would look like in these spaces.

Non-Archimedean norms

Let p be a prime number and a, b be natural numbers. We define the p-adic valuation as

$$\nu_p(a) := k \qquad \text{where} \qquad a = p^k c \qquad \text{for} \qquad p \nmid c. \tag{2}$$

We set $\nu_p(0) := \infty$. This defined on all rationals by

$$\nu_p(a/b) = \nu_p(a) - \nu_p(b). \tag{3}$$

We define the p-adic norm on $\mathbb{Q}$ by

$$|\cdot|_p := p^{-\nu_p(\cdot)} \tag{4}$$

1. Prove that $|\cdot|_p$ is a norm.
2. Prove that the natural numbers are bounded. Hence, $(Q, |\cdot|_p)$ does not satisfy the Archimedean property.
3. Prove the the strong triangle inequality i.e. for $x, y \in \mathbb{Q}$

$$|x+y|_p \leq \max\{|x|_p, |y|_p\}.$$

   Note for $d(x, y) = |x-y|_p$, the corresponding triangle inequality is

$$d(x, z) \leq \max\{d(x, y), d(y, z)\}.$$

4. Give an example of two real numbers x, y such that $|x+y| > \max\{|x|, |y|\}$.
5. Let $x \in \mathbb{Q}$ and $r \in \mathbb{R}_{>0}$. Use the strong triangle inequality to prove that for all $b \in B(x, r)$, $B(b, r) = B(x, r)$. (All points in a ball are at the center of the ball. Very weird!!)
6. Prove that open balls are closed. (Hint show that open balls contain their boundary points.)