1. Prove that

$$\left(1+\frac{1}{n}\right)^n \tag{1}$$

converges as $n \to \infty$.

2. Let $s_n \to L$ where $s_n, L \in \mathbb{R}$. Prove that

$$\frac{1}{n}\sum_{m=1}^{n} s_m \to L. \tag{2}$$

What about the converse?

3. Show that two equivalent norms induce equivalent metrics.

4. Prove that if A and B are open in $\mathbb{R}$ then $A \times B$ is open in $\mathbb{R}^2$ under the metric topology.

5. Show that for two equivalent metrics on the set E, they have the same topology. (Hint: Show that open balls with respect to one metric is open with respect to the other metric)

6. Prove that bounded sequences in $\mathbb{R}^n$ have a convergent subsequence.

7. Let C be a non-empty closed set in $\mathbb{R}^n$ and $x \in \mathbb{R}^n - C$. Prove there exists a $y \in C$ such that

$$||x - y|| = \inf\{||x - z|| : z \in C\}. \tag{3}$$

This point is then the closest point in C to x.

8. Give another definition of continuity prove it is equivalent to our current definition.

9. Prove that arbitrary intersections of compact sets are compact.

10. Are these functions uniformly continuous?
    (a) $f(x) = 1/x$ for $x \in (0, 1)$
    (b) $f(x) = 1/x$ for $x \geq 1$

1 Discussion

The metric topology on $\mathbb{R}^2$

1. Draw what an open ball of radius 1 looks like with respect to d_1, d_2,d_3 and d_∞.

2. See this animation. What do you notice about the open ball? How could you parameterize the boundary? (Hint: try to use cosine and sine)

Compact sets

Consider the set $\ell^1(\mathbb{R}) := \{(s_n)_{n=1}^{\infty} : \sum_{n=1}^{\infty} |s_n| \leq \infty\}$.

1. Prove that this is a metric space

2. Construct a closed and bounded set that is not sequentially compact.

3. Prove that the boundary of a compact set is compact.

4. For $a, b \in \mathbb{R}$, prove from first principles that $[a, b]$ is compact.

Continuous functions

1. Give another definition of continuity using compact sets and prove that it is equivalent to our definition.
2. Consider Thomae's function

$$f(x) = \begin{cases} \frac{1}{q} & \text{if } x \in \mathbb{Q} \text{ and } x = \frac{p}{q} \text{ in lowest terms } (q > 0) \\ 0 & \text{if } x \notin \mathbb{Q} \text{ (irrational)} \end{cases}$$

   (a) Prove it is continuous at irrationals.
   (b) Prove it is discontinuous at rationals.
   (c) Prove that $f(x+1) = f(x)$.