1 Homework

1. Suppose $(X, \mathcal{T})$ is a topological space. Let $K \subseteq X$ be compact, and let $f : K \to \mathbb{R}$ be continuous. Show that there exist $x_1, x_2 \in K$ such that for every $x \in K$,
$$f(x_1) \leq f(x) \leq f(x_2).$$
2. Show that if a continuous function $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(x) = 0$ for all $x \in \mathbb{Q}$, then $f(x) = 0$ for all $x \in \mathbb{R}$.
3. Give an example of a continuous function $f : \mathbb{R} \to \mathbb{R}$ and an open set U such that $f(U)$ is not open.
4. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Suppose that a sequence of functions $f_n(x)$ on $\mathbb{R}$ satisfy:
   (a) $0 \leq f_n(x) \leq 1$ for all $x \in \mathbb{R}$ and $n \geq 1$.
   (b) $f_n(x)$ is increasing in x for every $n \geq 1$.
   (c)
   $$\lim_{n\to\infty} f_n(x) = f(x)$$
   for each $x \in \mathbb{R}$.
   (d)
   $$\lim_{x\to-\infty} f(x) = 0, \qquad \lim_{x\to\infty} f(x) = 1.$$
   Show that $f_n(x) \to f(x)$ uniformly on $\mathbb{R}$.
5. Suppose $f : \mathbb{R} \to \mathbb{R}$ satisfies
$$|f(x) - f(y)| \leq M|x - y|$$
for all $x, y \in \mathbb{R}$. Such a function is called Lipschitz. Suppose that f is differentiable.
   (a) Show that f is continuous.
   (b) Show that
   $$|f'(x)| \leq M$$
   for every $x \in \mathbb{R}$.
   (c) Conversely, suppose that
   $$|f'(x)| \leq M$$
   for all $x \in \mathbb{R}$. Show that
   $$|f(x) - f(y)| \leq M|x - y|$$
   for all $x, y \in \mathbb{R}$.
6. Prove that
$$|\sin x - \sin y| \leq |x - y|.$$
7. Suppose f is k-times differentiable on an open interval I and
$$f^{(k)}(x) = 0$$
for all $x \in I$. What form does f have? Prove your claim.
8. Let
$$f(x) = \begin{cases} x, & \text{if } x \text{ is irrational,} \\ 0, & \text{if } x \text{ is rational.} \end{cases}$$
   (a) Calculate the upper and lower Darboux integrals of f on the interval $[0, b]$.
   (b) Give another example of a bounded function that is not Riemann integrable.
9. Show that if f is integrable on $[a, b]$, then f is integrable on every interval $[c, d] \subseteq [a, b]$.

Discussion

Riemann Integration

1. Let f be a bounded function on $[a,b]$. Prove that f is integrable if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every partition P of $[a,b]$ with $\operatorname{mesh}(P) \leq \delta$,
$$U(f,P) - L(f,P) < \varepsilon.$$
2. Prove that a bounded function f on $[a,b]$ is Darboux integrable if and only if it is Riemann integrable.

Cantor Set

1. Let
$$A_0 = [0,1], \qquad A_1 = [0, \tfrac{1}{3}] \cup [\tfrac{2}{3}, 1], \qquad A_2 = [0, \tfrac{1}{9}] \cup [\tfrac{2}{9}, \tfrac{1}{3}] \cup [\tfrac{2}{3}, \tfrac{7}{9}] \cup [\tfrac{8}{9}, 1].$$
For all n, define
$$A_{n+1} = \frac{A_n}{3} \cup \left(\frac{2}{3} + \frac{A_n}{3}\right).$$
Set
$$C = \bigcap_{n=0}^{\infty} A_n.$$
We call C the Cantor set. Show:
    (a) C is compact.
    (b) C has infinitely many points. (Hint: Consider the endpoints of the intervals.)
    (c) $\operatorname{int}(C) = \varnothing$.
    (d) C is totally disconnected; that is, given any $x, y \in C$, there are open sets U and V of C such that $C = U \cup V$, $U \cap V$, and $x \in U$ and $y \in V$.
    (e) Every point of C is a cluster point of C (sets with this property are called perfect).