Homework

1. Let $f : [a, b] \to \mathbb{R}$ be a strictly increasing function that is differentiable on (a, b). Suppose that $f'(x) \neq 0$ for all $x \in (a, b)$, and let $J = f([a, b])$.
   (a) Show that f^{-1} is differentiable on the interior of J.
   (b) Show that for every y in the interior of J,
   $$(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}.$$
2. Prove the following properties for the exponential function e^x:
   (a) The domain of e^x is $\mathbb{R}$.
   (b) $e^x > 0$ for all $x \in \mathbb{R}$.
   (c) $e^x e^y = e^{x+y}$ for all $x, y \in \mathbb{R}$.
3. Let $f : [-1, 1] \to \mathbb{R}$ be defined by
   $$f(x) := \begin{cases} x^2 \sin(1/x^2) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$
   (a) Show that f is differentiable on $(-1, 1)$.
   (b) Show that f' is not bounded on $(-1, 1)$, and thus f' is not Riemann integrable on $[-1, 1]$.
4. Let f be integrable on $[a, b]$, and suppose g is a function on $[a, b]$ such that $g(x) = f(x)$ except for finitely many $x \in [a, b]$. Show that
   $$\int_a^b f = \int_a^b g.$$
   Hint: Consider the case when f is identically equal to 0.
5. Prove the Mean Value Theorem for Integrals: If f is a continuous function on $[a, b]$, then there exists at least one $x \in (a, b)$ such that
   $$f(x) = \frac{1}{b-a} \int_a^b f(t)\, dt.$$

Discussion

1. Let $f : [0, 1] \to \mathbb{R}$ be defined by
   $$f(x) := \begin{cases} 1/q & \text{if } x = p/q \text{ with } \gcd(p, q) = 1, \\ 0 & \text{otherwise.} \end{cases}$$
   Show that f is Riemann integrable. Assume $f(0) = f(1) = 1$.
2. Give examples of the following:
   (a) A bounded function f where f^2 is integrable but f is not.
   (b) A bounded function f such that $|f|$ is integrable but f is not.
3. Let $f, g : [a, b] \to \mathbb{R}$ be integrable. Prove that $\max\{f(x), g(x)\}$ and $\min\{f(x), g(x)\}$ are integrable.